L. ROSENFELD

NUCLEAR FORCES

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I

EDITED BY

H. B. G. CASIMIR
DIRECTOR OF THE PHILIPS LABORATORIES, EINDHOVEN

AND

H. BRINKMAN
HEAD OF THE RESEARCH DEPARTMENT OF THE N.V. KEMA, ARNHEM

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BY

L. ROSENFELD

PROFESSOR OF THEORETICAL PHYSICS IN THE
UNIVERSITY OF MANCHESTER

FORMERLY PROFESSOR IN THE UNIVERSITY OF UTRECHT

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PART III

NUCLEAR MODELS AND SATURATION PROPERTIES ON CENTRAL FORCE HYPOTHESIS

CHAPTER IX

SURVEY OF NUCLEAR MODELS

9.0. Going over to the study of complex nuclei, we shall in the first place be concerned with the question as to which models can be used to describe the nuclear systems considered with more or less accuracy. We may state at the outset that no entirely adequate model has as yet been found; different approaches can be tried according to the point of view chosen and in each case the way is barred by mathematical difficulties. Nevertheless, many interesting results have been obtained, which we shall try to summarize in this Part.

9.1. Description of heavy nuclei

9.11. *Bohr's droplet model.* It has already been stressed (*2.22*) that heavier nuclei — in contrast with atoms — must be regarded as systems of closely coupled particles, in some way comparable to liquid droplets; the analogy, however, is very rough, since the mobility of the closely packed nucleons is a consequence of their unclassical zero-point motion (*2.3*). Additional confirmation of this picture is afforded by the discussion of nuclear reactions, initiated by BOHR [36]. The large probability of the capture of fast neutrons by heavy nuclei, together with the sharpness of the γ-rays emitted in such processes, forces us to conclude that the duration of the collision leading to capture is very much longer than the time which the neutron would take to cross the nucleus. In other words, the collision first leads to the formation of a *compound nucleus* of great stability, in a highly excited state; the ensuing emission of radiation, by which the compound nucleus goes over to its normal state, is a secondary process, occurring quite independently of the way in which the compound nucleus has been formed. This behaviour is easily understood just by realizing that the particles within the nucleus interact so strongly with each other and with the impinging neutron, that the energy of the latter is rapidly distributed among all nucleons; the possibility for any of these to escape from the compound nucleus is then conditioned by the occurrence of a casual fluctuation leading to a sufficient concentration of energy on that particular nucleon. These considerations clearly apply to any type of nuclear reaction, initiated by the impact of some elementary or compound particle (deuteron, α-particle) or a photon on a nucleus with emission of some other particle or of radiation. Such a reaction will take place in two independent stages, of which the first will be the formation of the compound nucleus; the second will be one of a number of competing modes of

disintegration of the compound nucleus or of radiative transitions to its ground state. From the study of nuclear reactions, much information has in particular been gathered about the distribution of the excited energy levels of compound nuclei; this distribution exhibits just the characteristic features to be expected for a system with close coupling: a rapid decrease of the mean distance between neighbouring levels with increasing excitation energy; at the same time, an increase of the breadth of the levels (owing to the various possibilities of transitions from the excited states), so that the level distribution becomes continuous (fig. *9.11*).

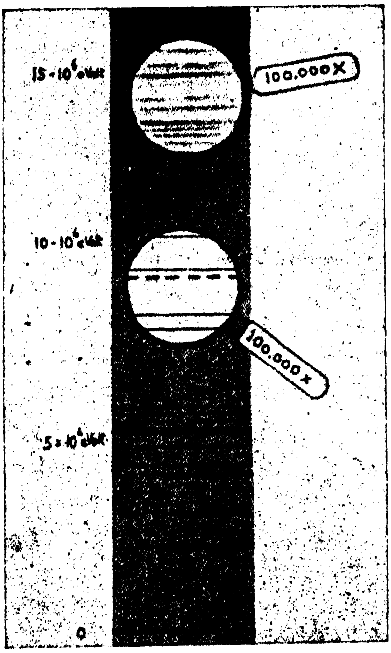

Fig. **9.11**. Schematic level system of a heavy nucleus. The dotted line in the lower "100.000 times magnifying glass" indicates the binding energy of a neutron in the nucleus, so that the levels above this line exhibit resonance with neutrons whose kinetic energy is given by the distance between the level considered and the dotted line.

Of course, such a system of closely coupled particles, as an actual nucleus proves to be, is rather unmanageable by present methods of quantum theory, since the usual approximation procedures of this theory have chiefly been developed for the treatment of loosely coupled atomic systems of electrons. On the other hand, methods derived from the analogy with a classical liquid droplet cannot claim more than qualitative validity (BOHR and KALCKAR [37]). In this connexion, *statistical considerations*, first advocated by FRENKEL [36], have proved most fruitful. According to this point of view, the motion of the constituent nucleons is compared with the thermal agitation of ordinary matter: to the nucleus in any excited state of energy E^* is attributed a temperature * T, related to E^* according to the rules of quantum statistics. Assuming e.g. the excited states of the

* We denote by T the temperature in the dynamic scale, i.e. the usual temperature in degrees times Boltzmann's constant k.

nucleus to consist of a superposition of harmonic vibrations, the frequencies of which are uniformly distributed, and using Planck's formula for the mean energy of a harmonic oscillator, one finds

$$E^* = \tfrac{1}{2}\gamma_A T^2, \tag{1}$$

the constant γ_A^{-1} representing $3/\pi^2$ times the average energy difference between two successive proper vibrations.

The entropy $S(E^*)$ of the nucleus in its state of energy E^* is simply related to the density $\varrho(E^*)$ of the energy levels at that energy by a formula of the type

$$\varrho(E^*) \sim e^{S(E^*)/k}, \tag{2}$$

while the well-known relation

$$\frac{1}{k}\frac{dS}{dE^*} = \frac{1}{T} \tag{3}$$

obtains between the derivative of the entropy and the temperature. Combining this last formula with the given relation, such as (1), between temperature and energy, we get an explicit expression for the function $S(E^*)$ or $\varrho(E^*)$. In our example, we get

$$\varrho(E^*) \sim e^{\sqrt{2\gamma_A E^*}}, \tag{4}$$

a formula * in good agreement with the empirically estimated level densities, if we take (WEISSKOPF [37])

$$\gamma_A^{-1} \approx 0{,}1 \text{ MeV}. \tag{5}$$

The emission of particles by the excited compound nucleus can be compared (FRENKEL [36], WEISSKOPF [37]) with an evaporation process, and for sufficiently high excitations, its probability can be estimated by well-known statistical arguments. To take only the simplest case of neutron emission, it is found, in particular, that the mean energy of the emitted neutrons will be $2T$, if T denotes the temperature of the nucleus left after the process. Owing to the relatively small number of constituent nucleons, this residual nucleus is considerably "cooled down" by the evaporation process: if, e.g., the excitation of the compound nucleus was 20 MeV, that of the residual nucleus will be only about 12 MeV. According to formulae (1) and (5), its temperature will be $\approx$ 1,5 MeV, and the average

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* This formula receives an interesting interpretation if we put the excitation energy E^* equal to n^* times the mean energy difference between successive proper vibrations; the resulting expression

$$\varrho(n^*) \sim e^{\pi\sqrt{2n^*/3}}$$

is just the asymptotic estimate of Hardy and Ramanujan for the number of partitions of the integer n^*. See also BOHR and KALCKAR [37], p. 34, and VAN LIER and UHLENBECK [37].

energy of the emitted neutrons about 3 MeV, thus very much smaller than the initial excitation energy. This striking consequence of the statistical conception is likewise in qualitative and rough quantitative agreement with experiment.

The above considerations will suffice to illustrate the kind of arguments which may be based on the droplet model and which in many cases have led to a very helpful global understanding of properties of heavy nuclei. The detailed form of nuclear interaction, however, clearly lies outside the scope of a description which treats nuclear matter as a continuous medium. We now turn our attention to another nuclear model which, though admittedly less rational than the droplet model, nevertheless in many respects gives a sufficiently adequate picture of heavy nuclei and allows of a much more elaborate treatment.

9.12. *The Fermi gas model.* It turns out that the behaviour of heavy nuclei is to a large extent so insensitive to the precise form of the interactions at close distances between the nucleons that it is correctly described — at any rate qualitatively — even if we begin by completely disregarding this interaction. To describe an excited nuclear state of temperature T, we thus start from an ideal gas of nuclear matter at this temperature, consisting of a given number A of nucleons enclosed in a volume of nuclear dimensions. In this initial approximation, the limitation of the system to a given volume is the only way in which the nuclear forces come into play. It may seem paradoxical that such an apparently inadequate picture of a nucleus should still possess fundamental features embodied in the liquid droplet model. The reason is that we are dealing with an assembly of particles obeying Fermi statistics in a strongly degenerate state: in fact, for all states of excitation commonly considered, the excitation energy (of 20 MeV, say) of an assembly of 50 or 100 particles of mean kinetic energy $\approx$ 20 MeV each (*2.3*) only means a slight departure from complete degeneracy. As a matter of fact, such a degenerate state resembles a liquid more than a gas, since the exclusion principle prevents two nucleons with the same charge and spin from occupying the same place within the nucleus; this has just the effect, even in the absence of any forces, of keeping the nucleons of the kind considered a certain mean distance apart, a feature which we have recognized as characteristic of nuclear structure (*2.22*).

In particular, it can be verified, as we shall see (*9.43*), that the degenerate gas of nucleons possesses the properties required to account for the course of nuclear reactions sketched above (*9.11*). For temperatures of a few MeV (corresponding to usual excitation energies), its viscosity is found to be so large that any vibration excited in it will be aperiodically damped, so that the excitation energy can only be kept in the form of heat. The heat conductivity is also very large: a local concentration of heat (such as produced by the impact of a particle) will rapidly spread over the whole nucleus.

Moreover, it is well-known (*9.42*) that the excitation energy of a degenerate Fermi gas depends on the temperature just by a relation of the form (1); the numerical value of the co-factor γ_A, calculated on this model for $A \approx 100$, is about twice as large as that corresponding to the empirical equation (5), but the discrepancy can be attributed to the influence of the nuclear forces (*9.42*).

If we now turn to the consideration of the ground state of a heavy nucleus, the Fermi gas model will again offer us a useful starting point, at least for a qualitative discussion; for the actual calculation of mass-defects, it is, of course, less satisfactory. The nuclear interactions (for which some analytical expression, e.g. of the type (*8.2*–1), is assumed) are introduced as perturbations, and the calculation of the total energy of the system may be improved by applying a variational procedure, in which the assumed nuclear radius is treated as the quantity to be varied. Calculations of this type have been carried out as far as the second approximation (*12.2*); from a quantitative point of view, their outcome is rather disappointing, but they lead to qualitative conclusions of considerable interest.

On the one hand, even the first approximation treatment suffices (together with a few additional considerations) for a general discussion of the saturation properties of the nuclear bonds, which enables us to complete the determination of the spin and charge dependence of the nuclear potential (Chapt. XI). On the other hand, the second approximation discloses a peculiar correlation between any four nucleons with different charges and spins, leading to the temporary formation of clusters resembling α-particles (*12.23*). This tendency to "α-clustering" illustrates from another side the inadequacy of the Fermi gas model: indeed, the considerable binding energy of such clusters is not properly taken into account in the computations based on the perturbation method. At the same time, it points to an approach to the problem of nuclear structure starting so to speak from the opposite end, viz. the so-called "α-particle model", in which the α-clusters (together with a few odd nucleons if necessary) are taken as fundamental units, held together by appropriate forces. For heavy nuclei, however, the latter model has no obvious advantage over the Fermi gas picture: its interest will first appear in the domain of lighter nuclei, which we shall now proceed to consider.

9.2. Description of light nuclei

9.21. *The quasi-atomic model.* As is well-known, we possess for the treatment of many-electron systems in the electric field of an atomic nucleus a powerful method initiated by Hartree and modified by Fock to take account of exchange interactions (*2.22*). Each particle is assumed to be acted upon by a potential field of suitable form, representing in a certain sense some average of the total field exerted on the particle by the other particles of the system: this potential involves no other dynamical variables

than those of the particle envisaged, while it depends on the other particles in a schematical way, through certain quantities treated as parameters. The states of the particles in this field are thus described by *individual* wave-functions, and the eigenfunctions of the total system, in the initial approximation, are linear combinations of Slater-determinants (*4.14*) built up of appropriate individual wave-functions. The eigenvalue problem can then be solved by the usual variation principle, the parameters of the potential field being varied and their "best" values determined.

The Hartree-Fock method has also been extensively applied to the treatment of light nuclei, in spite of the fact that it obviously loses any justification for such closely coupled systems; it is not surprising that the quantitative results obtained in this way are inconsistent and unreliable. The general scheme indeed differs essentially from the atomic case by the absence of any analogue of the central field of the atomic nucleus which constitutes a very considerable part of the Hartree field of the atom. The choice of the individual eigenfunctions of the nucleons is therefore to a large extent arbitrary; they are made to depend on certain adjustable parameters, which may be regarded as related to a fictitious average nuclear field. For instance, this field may be assumed to be quasi-elastic, so that the individual wave-functions of the nucleons have the form of (spatial) harmonic oscillator eigenfunctions, with arbitrary frequency parameters. The general picture of the nucleus corresponding to a Hartree-Fock procedure will be called the *quasi-atomic model.*

Formally, the quasi-atomic model and the Fermi gas model are very similar: they have in common the essential feature of starting from the consideration of individual states for the single nucleons; we may say that they belong to the class of *individual nuclear models*. In the Fermi gas picture, the individual wave-functions are simply plane waves confined to a region of nuclear dimensions; the parameters defining this region (nuclear radius, e.g.) play a similar role to that of the parameters of the Hartree-Fock wave-functions, viz. in schematically representing the influence of the nuclear forces in the initial approximation. It is clear that, so far as they go, the quasi-atomic model is better adapted to lighter nuclei, the Fermi gas model to heavier ones. But no more than qualitative indications can be expected from either.

9.22. *Nuclear models of the collective type.* Any adequate nuclear model must necessarily be of the *collective* type: i.e. the description of any state of the model, as embodied in the corresponding wave-function, must depend essentially on the variables of all the constituent nucleons. In some way the droplet model might be regarded as belonging to this class; but it is perhaps more correct to say that it falls outside the alternative: individual-collective, as it treats nuclear matter as a continuum. The most general and comprehensive form of collective nuclear model is the *resonating group model* (*13.3*). It attempts to analyze the states of nuclear systems

into a continually changing medley of all possible groupings of nucleons. In more technical language, the wave-function of any state of the system is set up as a superposition of all possible products of wave-functions representing smaller groups of nucleons. A workable approximation is then eventually obtained by judiciously weighing the probabilities of the various groupings and dropping the unlikely ones.

The *α-particle model* (*13.1*), for instance, falls under this general scheme: only α-clusters are retained as constituent groupings. It can be estimated that the stability of the temporary α-clusters is sufficient to justify such a treatment in many cases. Especially the study of the lightest nuclei which can be decomposed into a whole number of α-clusters ("α-nuclei") may be carried out in great detail by methods borrowed from the theory of molecules. As the number of α-clusters increases, their cohesion becomes loosened by the interplay of the nuclear forces and the periodicity due to α-clustering, which is so marked in the domain of light nuclei, gets more and more blurred out.

9.23. *The Wigner approximation.* The preceding approximate treatments of the nuclear problem do not involve any restrictive hypothesis regarding the law of nuclear force itself: they are rather concerned with some schematization of the structure of the system of nucleons assumed as a starting point for the computation of its binding energy or other properties. Another principle of approximation consists in leaving the structure *a priori* undetermined, but introducing simplifying assumptions about the nuclear interactions. One can then consider the most general collective model from the point of view of its transformation properties for the groups allowed by the assumed form of the nuclear potential; and by the methods of group theory, one can derive qualitative or semi-quantitative features of nuclear states which have a wide range of validity. Wigner, who has especially developed this method (*10.14*), has shown that a convenient starting point is obtained by first retaining only the ordinary and Majorana terms in the nuclear potential; the explicit spin and charge dependence, although by no means small, nevertheless appears to be of secondary importance for this kind of consideration. The Coulomb energy of the protons, however, will tend to disturb the symmetry properties essential for the application of the method and will therefore limit its validity for the heavier nuclei. The usefulness of *Wigner's approximation* for the study of light and intermediate nuclei is quite comparable with that of the analogous methods of group theory in atomic and molecular physics.

9.3. A table of nuclear models

It will be convenient to summarize in tabular form the conclusions of the preceding sections, the nuclear models discussed being classified from two distinct points of view: (*a*) according to their individual or collective character: (*b*) according as they are better adapted to the treatment of

lighter or heavier nuclei. Further, the position of Wigner's approximation has been indicated, the arrows pointing in the direction of increasing accuracy of the models.

9.3. *Nuclear models*

	Heavy nuclei	Light nuclei	
Continuous nuclear matter	Liquid droplet model		
Collective models	Resonating group model	α-particle model	Wigner approx.
Individual models	Fermi gas model	quasi-atomic model	

9.4. Elementary properties of Fermi gas model

9.40. In this section, we shall discuss some properties of a strongly degenerate ideal Fermi gas, in a form adapted to the case of a gas of nucleons enclosed in a region of nuclear dimensions. This forms the initial approximation of the Fermi gas model.

9.41. *Kinetic energy of ground state (absolute zero).* The ground state of the nuclear model under discussion corresponds, in the initial approximation, to the absolute zero of temperature. All the lowest individual states of neutrons and protons are occupied up to those with a maximum momentum given by the condition that the total number of occupied states be equal to the number of neutrons or protons, as the case may be. For each value of the momentum vector $\vec{p}$ there are two individual states, corresponding to the two possible orientations of the spin of the nucleon, and in the approximation here considered, these two states will have the same energy. On account of the pairing tendency of the nucleon spins (*3.12*), we shall expect that in the ground state as many spins as possible are saturated, i.e. that as many individual proton and neutron states as possible are doubly occupied. Since we are only interested in asymptotic formulae for large numbers of protons and neutrons, we shall suppose these numbers even, and accordingly all spins saturated in the ground state. For the same reason, the form assumed for the enclosure does not matter and we can take a cube having the same volume $\frac{4\pi}{3} r_0^3 A$ as a spherical nucleus of radius $R = r_0 A^{\frac{1}{3}}$ (*2.1*–1). The constant r_0 may be treated as a variation parameter, no assumption being made about its eventual dependence on the mass number A.

Consider now, e.g., the N neutrons of such a nucleus. The maximum value $p_m^{(N)}$ of the momentum of the occupied states is given by

$$p_m^{(N)} = \frac{(9\pi)^{\frac{1}{3}} \hbar}{2 r_0} \left(\frac{2}{g_s}\right)^{\frac{1}{3}} \left(\frac{N}{\frac{1}{2}A}\right)^{\frac{1}{3}} ; \qquad (1)$$

in this formula, g_s denotes the number of occupied states of given

momentum $\vec{p}$: in the ground state, therefore, $g_s = 2$; but the formula is also applicable to the case ($g_s = 1$) of all spins parallel, — a case which will occur in the course of a later discussion. The maximum kinetic energy $p_m^{(N)2}/2M$ we denote by $\mathsf{E}_m^{(N)}$, and we find that the average kinetic energy $\mathcal{K}^{(N)}$ may be written

$$\mathcal{K}^{(N)} = \tfrac{3}{5} N \mathsf{E}_m^{(N)}. \tag{2}$$

Similar formulae obtain for the proton gas.

It will be convenient to refer all quantities pertaining to any nucleus of N neutrons and Z protons to the corresponding ones for a fictitious nucleus, of dimensions such that $r_0 = \frac{1}{2} d$ (*2.22–7*), and consisting of equal numbers of protons and neutrons ($N = Z = \frac{1}{2} A$) with entirely saturated spins. The constitution of such a *standard heavy nucleus* corresponds to a complete neglect of Coulomb forces. In this case, the common value of $p_m^{(N)}$ and $p_m^{(Z)}$ is

$$p_m = (9\pi)^{\frac{1}{3}} \frac{\hbar}{e^2} m, \tag{3}$$

so that

$$\begin{aligned} K &= \tfrac{3}{5} \mathsf{E}_m = \frac{3}{10} (9\pi)^{\frac{2}{3}} \left(\frac{\hbar}{e^2}\right)^2 \frac{m}{M} \cdot m \\ &= 14{,}5 \text{ MeV}, \end{aligned} \tag{4}$$

$$\mathsf{E}_m = 24{,}2 \text{ MeV}. \tag{5}$$

In terms of these standard values, the general expression for the kinetic energy takes the form

$$\mathcal{K} = K \left(\frac{2}{g_s}\right)^{\frac{2}{3}} \left(\frac{\frac{1}{2} d}{r_0}\right)^2 \frac{1}{(\frac{1}{2} A)^{\frac{2}{3}}} \left[N^{\frac{5}{3}} + Z^{\frac{5}{3}}\right]; \tag{6}$$

this shows, in particular, that in the standard nucleus, the kinetic energy is proportional to the number of nucleons and has the value K per nucleon. More generally, if we express N and Z as $\frac{1}{2}(A \pm n)$ in terms of the mass number A and the neutron excess n, and if we expand $N^{\frac{5}{3}}$, $Z^{\frac{5}{3}}$ in powers of n/A, we get

$$\mathcal{K} = K \left(\frac{2}{g_s}\right)^{\frac{2}{3}} \left(\frac{\frac{1}{2} d}{r_0}\right)^2 A \left[1 + \frac{5}{9} \left(\frac{n}{A}\right)^2\right]. \tag{7}$$

The deviation of $\mathcal{K}$ from linearity in A thus always remains quite small for stable nuclei, since even for the heaviest ones, the ratio n/A is only * $\approx 0{,}2$.

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* From (*3.22–2*), we estimate, for large A,

$$\frac{n}{A} \approx \frac{\frac{3}{10} \frac{e^2}{r_0} A^{\frac{2}{3}}}{2 \varepsilon_1 \gamma + \frac{3}{10} \frac{e^2}{r_0} A^{\frac{2}{3}}} = \frac{A^{\frac{2}{3}}}{136 + A^{\frac{2}{3}}},$$

on account of (*2.21–5,6*).

The order of magnitude of the mean kinetic energy per particle, as given by (4), corresponds to expectation on general grounds (*2.3*). In this connexion, it should be noted that, for a given nuclear volume, the initial approximation here discussed clearly yields the smallest possible value for the average kinetic energy. The effect of the neglected nuclear interactions will indeed necessarily be to cause transitions of certain nucleons to unoccupied states of higher momentum, resulting in an increase in kinetic energy; a more general formal proof has been given by WATANABE [39].

As regards the analogy of nuclear matter with He II, alluded to in *2.3*, it should be noted that on the individual model an important difference arises with respect to the statistics obeyed by the constituent particles: Fermi statistics for the nucleons, Bose statistics for the He atoms. From this point of view, the analogy is much closer on the α-particle model.

9.42. *Kinetic energy of excited states.* As is well-known, the kinetic energy of an ideal Fermi gas at temperature T differs from the zero-point energy just calculated by a term proportional to T^2:

$$\mathsf{E}^* = \tfrac{1}{2}\gamma_A^0\, T^2; \tag{8}$$

The co-factor γ_A^0 has the general form

$$\gamma_A^0 = \frac{\pi^2}{2}\left[\frac{N}{\mathsf{E}_m^{(N)}} + \frac{Z}{\mathsf{E}_m^{(Z)}}\right] = \frac{\pi^2}{2\mathsf{E}_m}\left(\frac{g_s}{2}\right)^{\frac{2}{3}}\left(\frac{r_0}{\frac{1}{2}d}\right)^2 (\tfrac{1}{2}A)^{\frac{2}{3}}(N^{\frac{1}{3}} + Z^{\frac{1}{3}}), \tag{9}$$

which for the standard nucleus reduces to

$$\gamma_A^0 = \frac{\pi^2 A}{2\mathsf{E}_m} \qquad \text{(standard heavy nucleus).} \tag{10}$$

If we take the composition of the standard nucleus as representative of "nuclear matter", the specific heat per unit mass of the latter is accordingly

$$C_V = \frac{k}{MA}\,\gamma_A^0\, T = \frac{\pi^2}{2}\,\frac{k}{M\,\mathsf{E}_m}\, T. \tag{11}$$

For $A = 100$, the expression (10) gives, with (5), $(\gamma_A^0)^{-1} \approx 0{,}05$ MeV, i.e. about half the required value (*9.11*–5). As already stated (*9.12*), it can be seen (BARDEEN [37]) that the effect of the nuclear interactions will be just to decrease γ_A^0. In fact, the average force exerted on a nucleon in a specified individual state by all the others can be derived, as we shall see (*13.3*), from an effective potential which depends explicitly on the individual state in question. Thus the total energy of a nucleon in this state will have a further dependence on the momentum; and the (negative) potential energy will naturally decrease in absolute value with increasing momentum. This means that the derivative $d\mathsf{E}/dp$ of energy with respect to momentum will be larger than when the nuclear forces are entirely neglected. Therefore, the constant γ_A^0, which is proportional to the average number of individual states per unit energy, i.e. to $dp/d\mathsf{E}$, will have a smaller value than in the case of no force.

9.43. *Kinetic properties of nuclear matter.* Let us, as above, consider a standard heavy nucleus as a "lump" of nuclear matter of density *

$$\varrho = M \Big/ \frac{4\pi}{3} r_0^3 \tag{12}$$

fixed by the nuclear dimensions. We may then investigate the "mechanical" and "thermal" properties of this nuclear matter by the ordinary methods of kinetic theory, modified to apply to a strongly degenerate Fermi gas. The coefficients of viscosity η and of heat conductivity $\varkappa$ may be written in the familiar form

$$\begin{aligned} \eta &= a_\eta \cdot \tfrac{1}{3} \varrho\, v\, \Lambda_\eta \\ \varkappa &= a_\varkappa \cdot \tfrac{1}{3} \varrho\, \bar{v}\, C_V \Lambda_\varkappa ; \end{aligned} \tag{13}$$

in these formulae, $\bar{v}$ is the mean velocity of the nucleons, C_V the specific heat per unit mass, given by (11), the a's numerical factors and the Λ's mean free paths associated with the transport phenomena considered. In the case of a degenerate Fermi gas, the latter quantities show a marked increase with decreasing temperature, because the number of possible collisions is very much reduced on account of the exclusion principle. Indeed, at a temperature T the possible transitions are limited to the states of the "Maxwell tail" beyond the maximum zero-point energy E_m: in velocity-space, such states roughly fill a spherical shell of radius $v_m = \sqrt{2E_m/M}$ and thickness $\Delta v \approx \Delta E/p_m \sim T/\sqrt{2ME_m}$ (v_m, p_m are the maximum zero-point velocity and momentum, respectively). If $\mathcal{S}$ is the average collision cross-section corresponding to the transport process envisaged, we therefore have

$$\Lambda \sim \frac{1}{\dfrac{\varrho}{M} \cdot \mathcal{S} \cdot \left(\dfrac{\Delta v}{v_m}\right)^2} \sim \frac{M}{\varrho} \cdot \frac{1}{\mathcal{S}} \cdot \left(\frac{E_m}{T}\right)^2 . \tag{14}$$

Further using (11) and $\bar{v} \sim v_m$, we may put (13) in the form

$$\begin{aligned} \eta &= a_\eta \cdot \tfrac{1}{3} M^{\frac{1}{2}} E_m^{\frac{1}{2}} \frac{1}{\mathcal{S}_\eta} \left(\frac{E_m}{T}\right)^2 \\ \varkappa &= a_\varkappa \cdot \tfrac{1}{3} M^{-\frac{1}{2}} E_m^{\frac{1}{2}} \frac{\pi^2 k}{2} \frac{1}{\mathcal{S}_\varkappa} \frac{E_m}{T} . \end{aligned} \tag{15}$$

We see how the peculiar temperature dependence of the mean free paths determines a large increase of viscosity and heat conductivity at low temperatures.

The rigorous calculation of η and $\varkappa$, starting from the Maxwell-Boltz-

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* In this subsection, it will be convenient to take the velocity of light as unit of velocity.

mann equation, has been performed by TOMONAGA [38]. Averaging over the scattering angles and relative velocities yields

$$\begin{aligned} \mathcal{S}_\eta &= \int_0^1 \frac{x^5\,dx}{\sqrt{1-x^2}}\,[\bar{\mathcal{S}}^{(0)}(2\,v_m x) - \bar{\mathcal{S}}^{(2)}(2\,v_m x)] \\ \mathcal{S}_\varkappa &= \int_0^1 \frac{x^3\,dx}{\sqrt{1-x^2}}\,\bar{\mathcal{S}}^{(0)}(2\,v_m x)\,; \end{aligned} \tag{16}$$

in these formulae, $\bar{\mathcal{S}}^{(m)}(v)$ has the following meaning: if the differential collision cross-section for any nucleon pair is expanded in a series of zonal harmonics:

$$d\mathcal{S} = d\Omega \sum_l c_l(v)\, Y_l^0(\vartheta)\,, \tag{17}$$

one has

$$\bar{\mathcal{S}}^{(m)}(v) = \sum_l^{(\mathrm{even})} c_l(v) \sqrt{\frac{4\pi}{2l+1}}\, |\, Y_l^m(\vartheta = 0)|^2, \tag{18}$$

the summation extending over the even values of l only.

To estimate the coefficients $c_l(v)$, we have to express the differential cross-sections $d\mathcal{S}$ in terms of the phases according to the formulae (*6.21*–10) and (*7.11*–15) for collisions between neutron and proton and between like nucleons, respectively. According to (18), only even l-values in the expansion (17) of $d\mathcal{S}$, i.e. even powers of cos ϑ, are of importance. Putting therefore, as in (*6.21*–10), with the notation (*8.33*–21),

$${}^\sigma\mathcal{S}^{(\mathrm{even})} = \frac{\pi}{4k^2} \sum_l^{(\mathrm{even})} \sum_{l'}^{(\mathrm{even})} \sqrt{(2l+1)(2l'+1)}\; Y_l^0\, Y_{l'}^0\; {}^\sigma a^{(ll')} \tag{19}$$

and introducing an analogous definition of ${}^\sigma\mathcal{S}^{(\mathrm{odd})}$, we write the proton-neutron cross-section in the form

$$\begin{aligned} d\mathcal{S}_{p\text{-}n} = 2\,d\Omega\,[3\,({}^3\mathcal{S}^{(\mathrm{odd})} + {}^3\mathcal{S}^{(\mathrm{even})}) + {}^1\mathcal{S}^{(\mathrm{odd})} + {}^1\mathcal{S}^{(\mathrm{even})} \\ + \mathcal{S}\,(\text{odd powers of } \cos\vartheta)]; \end{aligned} \tag{20}$$

the factor 2 accounts for the two possible relative orientations of the spins of the colliding nucleons. Assuming charge independence of the nuclear forces (*8.1*) and neglecting the Coulomb interaction altogether, we can express the like-particle collision cross-section in a quite similar manner; formulae (*7.11*–14, 15) reduce in this case to

$$d\mathcal{S}_{p\text{-}p} = d\mathcal{S}_{n\text{-}n} = d\Omega \cdot 4\,(3 \cdot {}^3\mathcal{S}^{(\mathrm{odd})} + {}^1\mathcal{S}^{(\mathrm{even})}); \tag{21}$$

there is here no factor 2 because the formulae quoted already contained such a factor, expressing that in the proton-proton scattering experiments the recoil protons are also recorded. In all, we thus get

$$\begin{aligned} d\mathcal{S} = 2\,d\Omega\,[9 \cdot {}^3\mathcal{S}^{(\mathrm{odd})} + 3 \cdot {}^3\mathcal{S}^{(\mathrm{even})} + 3 \cdot {}^1\mathcal{S}^{(\mathrm{even})} + {}^1\mathcal{S}^{(\mathrm{odd})} \\ + \mathcal{S}\,(\text{odd powers of } \cos\vartheta)]. \end{aligned} \tag{22}$$

For the computation of the phases, Tomonaga assumes a nuclear potential of the form (*8.2*–1), $J(r)$ being a well of width $D = 2{,}2 \cdot 10^{-13}$ cm and depth $J = 30{,}35$ MeV. For the interaction parameters a, he makes the inadequate choice $a_0 = a_\tau = 0$ (*11.32*), whence, by (*8.2*–9, 12), ${}^3p = -\frac{1}{3}q$, ${}^1p = -3$, and he takes $q = \frac{3}{5}$. He further chooses for r_0 a value $0{,}53\, d$ somewhat larger than the one here adopted. But a choice of more likely numerical values would not appreciably alter the results.

Evaluation of the expressions (16), (18) leads to

$$\mathcal{S}_\eta \approx 10^{-26}\,\mathrm{cm}^2, \qquad \mathcal{S}_\varkappa \approx 2 \cdot 10^{-26}\,\mathrm{cm}^2. \tag{23}$$

As to the numerical factors $\alpha_\eta, \alpha_\varkappa$ one gets

$$\alpha_\eta = \frac{2\sqrt{2}}{5\pi^3}, \qquad \alpha_\varkappa = \frac{7\sqrt{2}}{16\pi^3}. \tag{24}$$

It should be noted that the collision radii derived from (23), as well as the particle wave-lengths, are of the same order of magnitude as the average distance between neighbouring particles, so that the kinetic methods employed strictly speaking lose their validity. Still, we may use the above results to get a general idea of the effects of the large viscosity and heat conductivity of nuclear matter. For this purpose, Tomonaga discusses two simple examples:

(1) An elastic vibration of circular frequency ν and mean energy $\bar{\mathsf{E}}$ undergoes, owing to the viscosity, a mean energy dissipation $\overline{d\mathsf{E}/dt}$, such that the corresponding time of relaxation τ_η is given by

$$\tau_\eta = \frac{\bar{\mathsf{E}}}{\overline{d\mathsf{E}/dt}} = \frac{3}{2}\frac{\varrho\, C^2}{\eta\, \nu^2}, \tag{25}$$

C being the velocity of sound (referred to the velocity of light). The latter quantity can easily be estimated (*12.21*–39) and one finds

$$C^2 \approx 0{,}81 \cdot 10^{-2}. \tag{26}$$

Inserting (15), (23), (24), (26) into (25), and remembering the definition of E_m, one gets

$$\tau_\eta\, \nu \approx 1{,}35 \cdot 2\pi \cdot \frac{T}{\mathsf{E}_m} \cdot \frac{T}{\hbar\nu}. \tag{27}$$

In an actual nucleus, the smallest vibration frequency is $\approx C/R$, which, for $R \approx 10^{-12}$ cm, means a quantum $\hbar\nu \approx 11$ MeV. Since, by (5), $\mathsf{E}_m \approx 24$ MeV and the nuclear temperatures for usual excitations do not exceed $T \approx 2$ MeV (say), we see that the vibration will be damped within a time considerably shorter than its period.

(2) Consider in unlimited nuclear matter a plane of discontinuity $(0 \mid T)$ of temperature; assume for simplicity that the heat conductivity and specific heat have constant values corresponding to the temperature $\frac{1}{2}T$.

The time $\tau_\varkappa$ necessary for the temperature variation at a distance l from the plane of discontinuity to reach the value $\frac{1}{4}T$ is readily found to be

$$\tau_\varkappa = \frac{1}{0{,}92} \cdot \frac{\varrho C_V l^2}{\varkappa}. \tag{28}$$

Comparing this with the time l/v_m in which a nucleon would on the average travel the same distance if there were no collisions, one gets, using (12), (15), (24),

$$\frac{\tau_\varkappa v_m}{l} \approx 28{,}2 \cdot \frac{\mathcal{S}_\varkappa l}{r_0^3} \cdot \left(\frac{T}{E_m}\right)^2; \tag{29}$$

with (23), $r_0 = 1{,}42 \cdot 10^{-13}$ cm, $l \approx R \approx 10^{-12}$ cm, this becomes

$$\frac{\tau_\varkappa v_m}{l} \approx 200 \left(\frac{T}{E_m}\right)^2, \tag{30}$$

i.e. for usual excitations $(T/E_m \approx 1/12)$,

$$\tau_\varkappa v_m/l \approx 1{,}4,$$

illustrating the great rapidity of heat propagation. For very high excitations, however, such as are produced by impact of fast nucleons (*12.3*), the local increase of temperature at the point of impact may reach $\approx$ 7 MeV (excitation energy $\approx$ 250 MeV) and $\tau_\varkappa v_m/l \approx 17{,}5$: in this case, a local "evaporation" may set in before the excitation has spread over the whole nucleus.

CHAPTER X

NUCLEAR MULTIPLETS

10.1. Classification of nuclear states

10.10. The stationary states of atomic nuclei can be classified according to the invariance properties of the Hamiltonian with respect to different groups of transformation in a way closely analogous to that followed in atomic theory; this point of view has been developed independently by HUND [37] and by WIGNER [37*a*, *b*]. The classification will naturally be the more detailed as more specific assumptions are made on the form of nuclear interaction. We shall proceed in order of increasing specialization of this form.

10.11. *General invariance properties.* In the first place, it follows quite generally from the invariance of nuclear structure with respect to spatial rotations, that the *total angular momentum* in any nuclear state is characterized by a quantum number J, taking either integral or half-integral values, in such a way that the square of the angular momentum is equal to $J(J+1)\,\hbar^2$. If the mass number A is even, the allowed values of J are $0, 1, 2, \ldots$; if A is odd, one may have $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

Likewise, the invariance of the Hamiltonian for a spatial symmetry transformation with respect to a point entails a definite *parity* of the nuclear terms. In such a transformation, the eigenfunction is multiplied by $+1$ or -1, and the term is then called even or odd, respectively.

10.12. *Charge multiplets.* There is a further quantum number of universal validity, expressing the invariance of the electric charge of a nuclear configuration. For this purpose, we can take an eigenvalue of the projection on the 3-axis of the total isotopic vector

$$\mathsf{T} = \tfrac{1}{2} \sum_i \boldsymbol{\tau}^{(i)}, \tag{1}$$

i.e.

$$\mathsf{T}_3 = \tfrac{1}{2} \sum_i \tau_3^{(i)}, \tag{2}$$

which can simply be interpreted as half the neutron excess of the configuration. The eigenvalues of T_3 are either integers or half integers, according as the mass number is even or odd.

The charge independence property of the nuclear forces allows us to proceed a step further: as long as we disregard Coulomb energy (and any possible deviation from charge independence of the proper nuclear interaction), we can avail ourselves of the invariance of the (approximate)

Hamiltonian for all rotations in isotopic space (*8.1*). The operator T^2 then assumes an eigenvalue $\mathsf{T}(\mathsf{T}+1)$, with integral or half-integral T (according as A is even or odd). To a given T correspond $2\mathsf{T}+1$ terms of the same energy, forming a (degenerate) *charge multiplet*; these terms belong to the eigenvalues of T_3, comprised between $-\mathsf{T}$ and $+\mathsf{T}$, i.e. to isobaric nuclei of different neutron excesses. Such a charge multiplet is split up into its $2\mathsf{T}+1$ components by the Coulomb interaction.

10.13. *Spin multiplets.* If we assume that the coupling between the spins and the translatory motions of the constituent nucleons is sufficiently small, we get the usual grouping of terms into *spin multiplets*, characterized by the quantum numbers L, S of total orbital momentum and total spin; L can have integral values, S integral or half-integral values, according as A is even or odd. The terms of the multiplet have total angular momenta comprised between $|L-S|$ and $L+S$; the multiplicity is $2S+1$.

Following Hund's proposal, nuclear terms belonging to a spin and charge multiplet can be denoted by symbols similar to those used in atomic spectroscopy, with a slight extension to take account of charge multiplicity: e.g. ${}^{53}P_0$ is a term of charge multiplicity 5 ($\mathsf{T}=2$), spin multiplicity 3 ($S=1$), orbital momentum $L=1$ and total angular momentum $J=0$.

In the quasi-atomic model (*9.21*), we may introduce individual nucleon states with definite principal and azimuthal quantum numbers n, l. All states with given n and l constitute an "orbital shell", which can be filled by $4(2l+1)$ nucleons. In any configuration built up of such states, the resultant orbital momentum L has a value derived from vector addition of the individual l's. A characteristic difference from the atomic case lies in the general tendency in nuclei for the stationary states with lower L-values to be the more tightly bound: the reason is that the main interaction between the nucleons, in contrast to that between the atomic electrons, is an attractive one.

10.14. *Wigner supermultiplets.* It has been especially emphasized by WIGNER [37*a*]* that it may be useful to consider a more comprehensive grouping of nuclear multiplets, obtained by neglecting all explicit dependence of the nuclear potential on spin and isotopic variables, i.e. by retaining only ordinary and Majorana forces ("Wigner approximation", *9.23*). The eigenfunctions of any (degenerate) spin and charge multiplet have the general form

$$\Psi(\vec{x}^{(i)}, \sigma_z^{(i)'}, \tau_3^{(i)'}) = \frac{1}{\sqrt{f}} \sum_{\varkappa=1}^{f} \Phi_\varkappa(\vec{x}^{(i)})\, \mathrm{X}_\varkappa(\sigma_z^{(i)'}, \tau_3^{(i)'}) \tag{3}$$

of a linear combination of products of (orthogonal and normalized) spatial wave-functions $\Phi_\varkappa$ and functions $\mathrm{X}_\varkappa$ of spin and isotopic variables; the linear combination being such as to make Ψ antisymmetric in all

* Minor corrections to this paper are to be found in WIGNER [37*b*], p. 951, footnote 9.

coordinates. On Wigner's approximation, the energies of a number of spin and charge multiplets will coincide, and the resulting *supermultiplet* will be entirely characterized by the symmetry properties of the spatial wave-functions $\Phi_\varkappa$ with respect to permutations of the nucleons, i.e. by their *symmetry character* *.

The symmetry properties in question are described by a *partition*

$$A_1 + A_2 + A_3 + A_4 = A \qquad (A_1 \geqslant A_2 \geqslant A_3 \geqslant A_4) \tag{4}$$

of the total number A of particles into four integers expressing that the $\Phi_\varkappa$ may be made antisymmetrical with respect to groups of at most $A_1, \ldots, A_4$ particles separately, or (what amounts to the same, since the total wave-function Ψ must be antisymmetric) that the corresponding factors $X_\varkappa$ are symmetrical with respect to groups of at most $A_1, \ldots, A_4$ nucleons. Symbolically, we denote the "antisymmetrized normal form" of $\Phi_\varkappa$ by $A\,(A_1 + A_2 + A_3 + A_4)$ and the "symmetrized normal form" of $X_\varkappa$ by $S\,(A_1 + A_2 + A_3 + A_4)$. The reason why the partition (4) contains at most 4 terms is that the $X_\varkappa$ obviously cannot be antisymmetrical in more than four particles: the *antisymmetrized* normal form of $X_\varkappa$ is therefore of the type

$$A\,(\underbrace{4 + \ldots + 4}_{k_4} + \underbrace{3 + \ldots + 3}_{k_3} + \underbrace{2 + \ldots + 2}_{k_2} + \underbrace{1 + \ldots + 1}_{k_1}),$$

which just implies that the symmetrized normal form has the character $S\,(A_1 + A_2 + A_3 + A_4)$, with

$$k_1 = A_1 - A_2, \quad k_2 = A_2 - A_3, \quad k_3 = A_3 - A_4, \quad k_4 = A_4. \tag{5}$$

The A's can simply be interpreted as the highest numbers of proton or

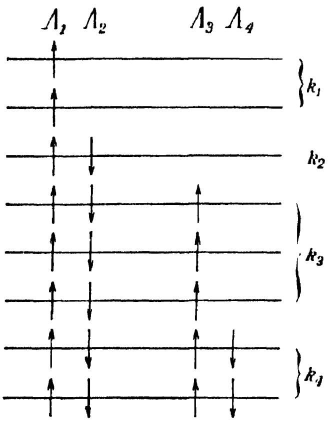

Fig. **10.14.** Configuration of symmetry character $A_1 + A_2 + A_3 + A_4$. The schematic representation used in this figure is especially adapted to individual nuclear models. The horizontal lines symbolize individual states with the same spatial wave-function; the arrows indicate the spin orientations with respect to a fixed spatial direction (z-components of spins). n = neutron states, p = proton states.

neutron states with definite spin orientation occurring in the supermultiplet; a configuration exemplifying this interpretation is represented by fig. *10.14*, which at the same time illustrates the connexion just utilized between

* A very clear and elementary account of the theory of symmetry characters has been given by HUND [27].

A- and S-symmetry characters, as well as the relations (5) between the k's and Λ's.

Let us introduce, besides T_3 defined by (2), the operators

$$S_z = \tfrac{1}{2} \sum_i \sigma_z^{(i)} \tag{6}$$

and

$$\mathsf{Y}_z = \tfrac{1}{2} \sum_i \tau_3^{(i)} \sigma_z^{(i)}, \tag{7}$$

representing, respectively, the z-component of the total spin and the difference of the z-components of the total neutron spin and the total proton spin. With the choice of the Λ's indicated by fig. *10.14*, the quantities

$$\begin{aligned} P &= \tfrac{1}{2}(\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4) \\ P' &= \tfrac{1}{2}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4) \\ P'' &= \tfrac{1}{2}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4) \end{aligned} \tag{8}$$

have the following meaning: P is the highest value of T_3 (and, therefore, of T) occurring in the supermultiplet, P' is the highest value of S_z (and of S) compatible with the value P of T_3, while P'' represents the highest value that Y_z can take when the other two operators have the respective values P and P'. Since, however, the correspondence between the Λ's and the spin orientations of the proton and neutron states can be established in different ways, by permuting the eigenvalues $+1$ and -1 of τ_3 and σ_z, a more general interpretation of the quantum numbers P, P', P'' is possible: P represents the highest value of any one of the quantities T_3, S_z, Y_z occurring in the supermultiplet; P' is then the highest value of a second one of the three quantities compatible with the value P of the first, and P'' the highest value of the third compatible with the values P, P' of the other two.

On account of the definition (4) of the Λ's, one has (WIGNER [37*b*])

$$\tfrac{1}{2} A \geqslant P \geqslant P' \geqslant |P''|. \tag{9}$$

It further follows from (4) and (8) that the numbers P, P', P'' are integers or half integers according as A is even or odd, and that

$$P + P' + P'' \text{ is } \begin{cases} \text{even if } A = 4a \\ \text{odd if } A = 4a + 2 \end{cases} \tag{10}$$

(since $P + P' + P'' = 2\Lambda_1 - \tfrac{1}{2}A$). Clearly the set P, P', P'' gives, together with A, a characterization of the supermultiplet equivalent to that expressed by the Λ's. It will be noted that the values of P, P', P'' given by (8) remain unchanged when the Λ's all change by the same amount. Physically, this change means adding to the original nucleus a certain number of sets of four nucleons with different charges and spins *; formally, as shown by (5), the antisymmetrized normal form of the $\mathsf{X}_\varkappa$ contains an additional number of sets of four coordinates with respect to which it is antisymmetrical.

* Such sets of nucleons might loosely be described as "α-particles".

The substates of a supermultiplet belong to eigenvalues of S_z, T_3 and Y_z, but — in contrast to the well-known case of the ordinary spin multiplets — such a set of eigenvalues may belong to more than one substate. These substates are in fact split up by the forces depending on the spin and isotopic variables into different spin and charge multiplets, generally giving rise to several states with the same set of eigenvalues of S_z and T_3. (In this splitting, states with different values of Y_z will in general combine, so that Y_z ceases to furnish a quantum number.)

10.2. Stationary states of light nuclei

10.20. The quantum numbers and energies of the stationary states to be expected for light nuclei can be determined by means of the *quasi-atomic model*. Quantitative energy estimates can hardly be trusted, but the resulting order of the levels, i.e. the assignment of L, S, T values to the successive terms, might still be expected to come out right. The ground states and excited terms have been found in this way for nuclear configurations corresponding to the successive completion of the first s and p orbital shells (as well as for some configurations in which also the first d shell is partly filled up). These results are summarized in the following tables, compiled from HUND's [37] paper; similar results have been obtained independently by WIGNER [37*a*] and his collaborators (FEENBERG and WIGNER [37*a*], FEENBERG and PHILLIPS [37*b*]). We have only retained the enumeration of the possible stationary states in their probable order of excitation; for the (unreliable) energy computations, the reader is referred to the papers quoted.

From the present point of view, the nuclei are conveniently divided into three classes, according as A is of the form $4a \pm 1$, $4a + 2$ or $4a$. In the first column of each table the uncompleted shell is indicated by its symbol s, p, d with the number of nucleons in this shell as an exponent; e.g. d^{10} represents the configuration $s^4p^{12}d^{10}$; configurations with k and $f-k$ nucleons in a shell comprising f individual states have the same types and arrangement of levels (the latter configuration has k "holes" in the last shell). The second column gives the Wigner supermultiplets (P, P', P''), the ground state lowest. In the third column are found the corresponding partitions of the X_r; the Λ's composing the partition $S(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)$ are only defined *modulo* 1, and there is a corresponding indeterminacy in the number of 4's occurring in the reciprocal symmetry character $A(4+\ldots)$. The subsequent columns contain, for the various possible values of the (absolute) neutron excess $|n|$, the possible types of levels, also arranged in order of (probable) excitation; levels connected by a $=$-sign coincide on Wigner's approximation.

Auxiliary tables give probable attributions of definite configurations to the known light nuclei of each class; the justification for such attributions will be found in *10.32*. Finally, a small table is added to bring out the regularities concerning the ground states of the nuclei considered; for odd nuclei, it has been assumed that the spin dependent forces depress the triplet term (*11.41*).

10.21. *Nuclei of odd mass number*

Shell filling	Wigner super-multiplet	Partitions	$\lvert n \rvert = 1$	$\lvert n \rvert = 3$	$\lvert n \rvert = 5$	Partitions	Wigner super-multiplet	Shell filling
s	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1)$, $S(1)$	^{22}S			$A(3)$, $S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	s^3
p	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1)$, $S(1)$	^{22}P			$A(3)$, $S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	p^{11}
p^3	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}$	$A(1+1+1)$, $S(3)$	$^{44}S = {}^{22}S$	^{22}S		$A(3+3+3)$, $S(3+3+3)$	$\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}$	p^9
	$\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$	$A(2+1)$, $S(2+1)$	$^{42}D = {}^{24}D = {}^{22}D$ $^{42}P = {}^{24}P = {}^{22}P$	^{42}D ^{42}P		$A(3+2)$, $S(2+2+1)$	$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$	
	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$A(3)$, $S(1+1+1)$	^{22}F ^{22}P			$A(1)$, $S(1)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
p^5	$\frac{5}{2}, \frac{1}{2}, \frac{1}{2}$	$A(2+2+1)$, $S(3+2)$	$^{62}P = {}^{26}P = {}^{44}P = {}^{42}P$ $= {}^{24}P = {}^{22}P$	$^{62}P = {}^{44}P = {}^{42}P$	^{62}P	$A(3+2+2)$, $S(3+3+1)$	$\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}$	p^7
	$\frac{3}{2}, \frac{3}{2}, \frac{1}{2}$	$A(3+1+1)$, $S(3+1+1)$	$^{44}D = {}^{42}D = {}^{24}D = {}^{22}D$ $^{44}S = {}^{42}S = {}^{24}S = {}^{22}S$	$^{44}D = {}^{42}D$ $^{44}S = {}^{42}S$		$A(3+3+1)$, $S(3+2+2)$	$\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}$	
	$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$	$A(3+2)$, $S(2+2+1)$	$^{42}F = {}^{24}F = {}^{22}F$ $^{42}D = {}^{24}D = {}^{22}D$ $^{42}P = {}^{24}P = {}^{22}P$	^{42}F ^{42}D ^{42}P		$A(2+1)$, $S(2+1)$	$\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$	
	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(4+1)$, $S(2+1+1+1)$	^{22}G ^{22}F ^{22}D ^{22}P			$A(3)$, $S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	
d	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$A(1)$, $S(1)$	^{22}D			$A(3)$, $S(1+1+1)$	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	d^{19}

Parity: when s or d shell is filling, all terms even; when p shell is filling, all terms odd.

10.211. *Light nuclei of odd mass number*

	$n =$				
	-1	$+1$	-1	$+1$	
s	${}^{17}_{9}F$	${}^{17}_{8}O$	${}^{3}_{2}He$ ${}^{19}_{10}Ne$	${}^{3}_{1}H$ ${}^{19}_{9}F$	s^3
p		${}^{5}_{2}He$	${}^{15}_{8}O$	${}^{15}_{7}N$	p^{11}
p^3	${}^{7}_{4}Be$	${}^{7}_{3}Li$	${}^{13}_{7}N$	${}^{13}_{6}C$	p^9
p^5	${}^{9}_{5}B$	${}^{9}_{4}Be$	${}^{11}_{6}C$	${}^{11}_{5}B$	p^7
d	${}^{21}_{11}Na$	${}^{21}_{10}Ne$	$({}^{39}_{20}Ca)$	${}^{39}_{19}K$	d^{19}

10.221. *Light nuclei of mass number $A = 4a+2$*

	$n =$				
	0	2	0	2	
s^2	${}^{2}_{1}H$ ${}^{18}_{9}F$	${}^{18}_{8}O$			
p^2	${}^{6}_{3}Li$	${}^{6}_{2}He$	${}^{14}_{7}N$	${}^{14}_{6}C$	p^{10}
p^6	${}^{10}_{5}B$	${}^{10}_{4}Be$			

10.22. *Nuclei of mass number* $A = 4a + 2$

Shell filling	Wigner super-multiplet	Partitions		$n = 0$ (odd nuclei)	$\|n\| = 2$ (even nuclei)	$\|n\| = 4$ (odd nuclei)	$\|n\| = 6$ (even nuclei)
s^2	1, 0, 0	$A(2)$,	$S(1+1)$	${}^{31}S = {}^{13}S$	${}^{31}S$		
p^2, p^{10}	1, 1, 1	$A(1+1)$,	$S(2)$	${}^{11}P = {}^{33}P$	${}^{33}P$		
	1, 0, 0	$A(2)$,	$S(1+1)$	${}^{31}D = {}^{13}D$ ${}^{31}S = {}^{13}S$	${}^{31}D$ ${}^{31}S$		
p^6	3, 0, 0	$A(2+2+2)$,	$S(3+3)$	${}^{71}S = {}^{17}S = {}^{53}S = {}^{35}S$ $= {}^{31}S = {}^{13}S$	${}^{71}S = {}^{53}S = {}^{35}S = {}^{31}S$	${}^{71}S = {}^{53}S$	${}^{71}S$
	2, 1, 0	$A(3+2+1)$,	$S(3+2+1)$	${}^{53}D = {}^{35}D = {}^{51}D = {}^{15}D$ $= {}^{33}DD = {}^{31}D = {}^{13}D$ ${}^{53}P = {}^{35}P = {}^{51}P = {}^{15}P$ $= {}^{33}PP = {}^{31}P = {}^{13}P$	${}^{53}D = {}^{35}D = {}^{51}D$ $= {}^{33}DD = {}^{31}D$ ${}^{53}P = {}^{35}P = {}^{51}P$ $= {}^{33}PP = {}^{31}P$	${}^{53}D = {}^{51}D$ ${}^{53}P = {}^{51}P$	
	1, 1, –1	$A(3+3)$,	$S(2+2+2)$	${}^{33}FF = {}^{11}FF$ ${}^{33}PP = {}^{11}PP$	${}^{33}FF$ ${}^{33}PP$		
	1, 1, 1	$A(4+1+1)$,	$S(3+1+1+1)$	${}^{33}FF = {}^{11}FF$ ${}^{33}PP = {}^{11}PP$	${}^{33}FF$ ${}^{33}PP$		
	1, 0, 0	$A(4+2)$,	$S(2+2+1+1)$	${}^{31}G = {}^{13}G$ ${}^{31}F = {}^{13}F$ ${}^{31}D = {}^{13}D$ ${}^{31}D = {}^{13}D$ ${}^{31}S = {}^{13}S$	${}^{31}G$ ${}^{31}F$ ${}^{31}D$ ${}^{31}D$ ${}^{31}S$		

Parity: all terms e v e n.

10.23. *Nuclei of mass number $A = 4a$*

Shell filling	Wigner super-multiplet	Partitions	$n = 0$ (even nuclei)	$\lvert n \rvert = 2$ (odd nuclei)	$\lvert n \rvert = 4$ (even nuclei)
s^4	0, 0, 0	$A(4)$, $S(1+1+1+1)$	^{11}S		
p^4, p^8	2, 1, 1	$A(2+1+1)$, $S(3+1)$	$^{53}P = {}^{35}P = {}^{33}P = {}^{31}P = {}^{13}P$	$^{53}P = {}^{35}P = {}^{33}P = {}^{31}P$	^{53}P
	2, 0, 0	$A(2+2)$, $S(2+2)$	$^{51}D = {}^{15}D = {}^{33}D = {}^{11}D$ $^{51}S = {}^{15}S = {}^{33}S = {}^{11}S$	$^{51}D = {}^{33}D$ $^{51}S = {}^{33}D$	^{51}D ^{51}S
	1, 1, 0	$A(3+1)$, $S(2+1+1)$	$^{33}F = {}^{31}F = {}^{13}F$ $^{33}D = {}^{31}D = {}^{13}D$ $^{33}P = {}^{31}P = {}^{13}P$	$^{33}F = {}^{31}F$ $^{33}D = {}^{31}D$ $^{33}P = {}^{31}P$	
	0, 0, 0	$A(4)$, $S(1+1+1+1)$	^{11}G ^{11}D ^{11}S		
p^{12}	0, 0, 0	$A(4)$, $S(1+1+1+1)$	^{11}S		
d^{20}	0, 0, 0	$A(4)$, $S(1+1+1+1)$	^{11}S		

Parity: all terms e v e n.

10.231. *Light nuclei of mass number $A = 4a$*

	$n =$ 0	2	0	2	
s^4	$^{4}_{2}He$ $^{20}_{10}Ne$				
p^4 p^{12}	$^{8}_{4}Be$ $^{16}_{8}O$	$^{8}_{3}Li$	$^{12}_{6}C$	$^{12}_{5}B$	p^8
d^{20}	$^{40}_{20}Ca$				

10.24. *Ground states of light nuclei*

Type of nucleus		Ground state
odd-mass nuclei	s shell filling	^2S
	p shell filling	^2P
even nuclei		^1S
odd nuclei	$n = 0$	^3S
	$n \neq 0$	^3P

10.3. Energies of nuclear states

10.31. *Nuclear potential energy in a supermultiplet state.* The nuclear interaction operator which underlies Wigner's approximation is of the form (*4.341*)

$$\mathcal{V}_W = \tfrac{1}{2} \sum_{i,k} J(\vec{x}^{(i)} - \vec{x}^{(k)}) (W + M P_x^{(ik)}). \tag{1}$$

Since it does not affect the factors $X_\varkappa$ of the eigenfunction (*10.14*–3), its expectation value in the corresponding state may be written

$$\begin{aligned} \operatorname{av} \mathcal{V}_W = \frac{1}{f} \sum_{\varkappa=1}^{f} \tfrac{1}{2} \sum_{i,k} \int \Phi_\varkappa^* (\vec{x}^{(1)}, \ldots, \vec{x}^{(A)})\, J(\vec{x}^{(i)} - \vec{x}^{(k)}) \\ \cdot (W + M P_x^{(ik)})\, \Phi_\varkappa (\vec{x}^{(1)}, \ldots, \vec{x}^{(A)})\, dv^{(1)} \ldots dv^{(A)}. \end{aligned} \tag{2}$$

The evaluation of this expression may be effected by a method due to WIGNER [*37b*]. It is based on the fundamental property of the functions $\Phi_\varkappa$ of given symmetry character to transform linearly among themselves when a permutation R is performed on the space coordinates:

$$R\, \Phi_\varkappa (\vec{x}^{(l)}) \equiv \Phi_\varkappa (R\, \vec{x}^{(l)}) = \sum_{\lambda=1}^{f} D_{\varkappa\lambda}(R)\, \Phi_\lambda (\vec{x}^{(l)}); \tag{3}$$

the unitary matrices $D(R)$ are determined (apart from a common transformation $D \to S^{-1}DS$) by the symmetry character of the $\Phi_\varkappa$: — in the language of group theory they form an irreducible representation of degree f of the symmetric group. The expectation value of any operator $\mathcal{W} = \frac{1}{2} \sum \mathcal{W}^{(ik)}$, with $\mathcal{W}^{(ik)} \equiv \mathcal{W}(\vec{x}^{(i)}, \vec{x}^{(k)})$, can then be expressed in terms of $\mathcal{W}^{(12)}$ only, as follows:

$$\begin{aligned} \operatorname{av} \mathcal{W} &= \frac{1}{f} \sum_{\varkappa=1}^{f} \tfrac{1}{2} \sum_{i,k} \int \Phi_\varkappa^* (P_x^{(i1)} P_x^{(k2)} \vec{x}^{(l)})\, \mathcal{W}^{(12)}\, \Phi_\varkappa (P_x^{(i1)} P_x^{(k2)} \vec{x}^{(l)})\, dv^{(1)} \ldots dv^{(A)} \\ &= \frac{1}{f} \sum_{\varkappa,\lambda,\mu=1}^{f} \tfrac{1}{2} \sum_{i,k} D_{\varkappa\lambda}^* (P_x^{(i1)} P_x^{(k2)})\, D_{\varkappa\mu} (P_x^{(i1)} P_x^{(k2)}) \int \Phi_\lambda^* (\vec{x}^{(l)})\, \mathcal{W}^{(12)}\, \Phi_\mu (\vec{x}^{(l)})\, dv^{(1)} \ldots dv^{(A)}; \end{aligned}$$

or, since $\sum_\varkappa D_{\varkappa\lambda}^* D_{\varkappa\mu} = \delta_{\lambda\mu}$,

$$\operatorname{av} \mathcal{W} = \frac{A(A-1)}{2} \frac{1}{f} \sum_{\varkappa=1}^{f} \int \Phi_\varkappa^* (\vec{x}^{(l)})\, \mathcal{W}^{(12)}\, \Phi_\varkappa (\vec{x}^{(l)})\, dv^{(1)} \ldots dv^{(A)}. \tag{4}$$

We may now make use of the above-mentioned arbitrariness in the representation D to make all $\Phi_\varkappa$ either symmetric or antisymmetric in $\vec{x}^{(1)}, \vec{x}^{(2)}$, while retaining their orthogonality; if the first s functions $\Phi_\varkappa$ are symmetric in $\vec{x}^{(1)}, \vec{x}^{(2)}$ and the others antisymmetric, the matrix $D(P_x^{(12)})$ will be diagonal, the first s elements being $= +1$ and the others $= -1$, so that s is defined in an invariant way by

$$\operatorname{tr} D(P_x^{(12)}) = 2s - f. \tag{5}$$

(The trace of the matrix $D(P_x^{(12)})$ is called in group theory the "character" of the operation $P_x^{(12)}$ in the representation in question.) Let us denote by

$$g_s(\vec{x}^{(1)}, \vec{x}^{(2)}) = \frac{1}{s} \sum_{\varkappa=1}^{s} \int |\Phi_\varkappa|^2 \, dv^{(3)} \, dv^{(4)} \ldots dv^{(A)}$$
$$g_a(\vec{x}^{(1)}, \vec{x}^{(2)}) = \frac{1}{f-s} \sum_{\varkappa=s+1}^{f} \int |\Phi_\varkappa|^2 \, dv^{(3)} \, dv^{(4)} \ldots dv^{(A)} \tag{6}$$

the probabilities for two symmetrically or antisymmetrically coupled particles to be at a distance $|\vec{x}^{(1)} - \vec{x}^{(2)}|$ from each other.

Turning back to (2), we may write it successively, using (4), (3) and (6),

$$\text{av } V_W = \frac{A(A-1)}{2} \frac{1}{f} \sum_{\varkappa=1}^{f} \int \Phi_\varkappa^* \, J(\vec{x}^{(1)} - \vec{x}^{(2)}) \, (W + M P_x^{(12)}) \, \Phi_\varkappa \, dv^{(1)} \ldots dv^{(A)}$$
$$= \frac{A(A-1)}{2} \frac{1}{f} \sum_{\varkappa=1}^{f} \int \Phi_\varkappa^* \, J(\vec{x}^{(1)} - \vec{x}^{(2)}) \, [W + M D_{\varkappa\varkappa}(P_x^{(12)})] \, \Phi_\varkappa \, dv^{(1)} \ldots dv^{(A)} \tag{7}$$
$$= \frac{A(A-1)}{2} \frac{1}{f} \left[(W+M) s \int g_s \, J \, dv^{(1)} \, dv^{(2)} + (W-M)(f-s) \int g_a \, J \, dv^{(1)} \, dv^{(2)} \right].$$

Putting

$$\mathcal{L}^{\pm} = \tfrac{1}{2} \int (g_s \pm g_a) \, J \, dv^{(1)} \, dv^{(2)}, \tag{8}$$

and taking account of (5), we finally get

$$\text{av } V_W = \frac{A(A-1)}{2} (W \mathcal{L}^+ + M \mathcal{L}^-) - \Xi (W \mathcal{L}^- + M \mathcal{L}^+)$$

with

$$\Xi = -\frac{A(A-1)}{2} \frac{1}{f} \operatorname{tr} D(P_x^{(12)}). \tag{9}$$

The interest of this formula lies in the fact that the coefficient Ξ is entirely determined by the symmetry character of the supermultiplet and can thus be calculated without knowing the eigenfunctions.

The integrals $\mathcal{L}^{\pm}$ cannot be evaluated on general lines, but they may reasonably be expected to vary smoothly with the mass number and the supermultiplet considered: the rapid irregular variations of nuclear energies with mass number should then be accounted for by Ξ; we shall see presently how this conception has been put to the test of observation. If we assume $J(r)$ to be positive for all values of r, and to vary sufficiently smoothly with r, both integrals $\mathcal{L}^{\pm}$ will be positive. The bulk of the integrals is contributed by an interval of distances $|\vec{x}^{(1)} - \vec{x}^{(2)}|$ of the order of the range of force, which for large nuclei is small compared with the nuclear radius. The g's being in that case about inversely proportional to

the square of the nuclear volume, the $\mathcal{L}$'s will be inversely proportional to this volume: in other words, for large values of A, the $\mathcal{L}$'s become inversely proportional to A.

The explicit expression for Ξ in terms of the quantum numbers P, P', P'' of the supermultiplet is, of course, immediately given by group theory. But it can also be derived in a quite elementary way. In the first place, we may treat the problem on the fictitious assumption that the range of the nuclear force is so large, compared with nuclear dimensions, that the variation of $\mathcal{J}(r)$ over the volume of the nucleus can be neglected: we shall call this the *long range fiction*. In this case, $\mathcal{L}^+ = \mathcal{J}$ and $\mathcal{L}^- = 0$, so that (9) becomes

$$\text{av}\ \mathcal{V}_{W,\,\text{long range}} = [\tfrac{1}{2} A (A-1)\, W - \Xi\, M]\, \mathcal{J}. \tag{10}$$

The interpretation of the first term of this formula is evident; the second term shows us that we shall get an expression for Ξ by calculating in a more direct way the expectation value of the nuclear potential energy for a pure Majorana interaction of long range. In fact,

$$\Xi = -\,\text{av}\ \tfrac{1}{2} \sum_{i,k} P_x^{(ik)}, \tag{11}$$

and simple symmetry considerations (HUND [37]) lead to the following result: Let n_a be the number of antisymmetrically coupled pairs of coordinates in the antisymmetrized normal form of the spatial wave-functions, and n_s the number of symmetrically coupled pairs in their symmetrized normal form. We may then write:

$$\Xi = n_a - n_s. \tag{12}$$

We now go a step further in the schematization of the problem by assuming an individual model and choosing a specified stationary state of the given symmetry character; that corresponding to the configuration represented by fig. *10.14* is especially suited to our purpose, because it permits an easy evaluation of the numbers n_a, n_s: the former is the number of pairs of like nucleons with parallel spins, while the latter is the number of *bonds*, this term denoting a pair of nucleons which belong to the same spatial wave-function. This result can be verified directly from (11) (INGLIS and YOUNG [37]) by setting up the total wave-function which describes the stationary state in question as a Slater determinant (*4.14*) and by remembering that (*4.341*–32) $P_x = -P_\sigma P_\tau$. From fig. *10.14* we read off

$$\begin{aligned} n_a &= \sum_{i=1}^{4} \tfrac{1}{2} \Lambda_i (\Lambda_i - 1) \\ n_s &= 6\, k_4 + 3 k_3 + k_2 \\ &= 3\, \Lambda_4 + 2\, \Lambda_3 + \Lambda_2, \end{aligned} \tag{13}$$

by (*10.14*–5), so that

$$\Xi = \tfrac{1}{2}\,[\Lambda_1(\Lambda_1-1) + \Lambda_2(\Lambda_2-3) + \Lambda_3(\Lambda_3-5) + \Lambda_4(\Lambda_4-7)]. \tag{14}$$

Using (*10.14*–4, 8), we may also write

$$\Xi = \tfrac{1}{2}\,[\tfrac{1}{2}A(\tfrac{1}{2}A-8) + P(P+4) + P'(P'+2) + P''^2] \tag{15}$$

or

$$\begin{aligned} \Xi &= \tfrac{1}{8}A^2 - 2A - \tfrac{5}{2} + \Xi' \\ \text{with}\qquad \Xi' &= \tfrac{1}{2}\,[(P+2)^2 + (P'+1)^2 + P''^2]. \end{aligned} \tag{16}$$

10.311. *Supermultiplet quantum numbers of ground state.* As we shall see in the following Chapter (*11.33*), we have reason to suppose that the parameters W and M in formula (1) satisfy the inequalities

$$-M > W > 0 \tag{17}$$

Since, on the other hand, the $\mathcal{L}$'s are positive and $\mathcal{L}^+ > \mathcal{L}^-$, the coefficient $-(W\mathcal{L}^- + M\mathcal{L}^+)$ of Ξ in formula (9) will be positive. The maximum binding energy will therefore correspond to the minimum value of Ξ or, by (16), of Ξ'; i.e. the ground state will belong to the supermultiplet with the smallest possible quantum numbers P, P', $|P''|$.

The neutron excess n being given, we therefore take

$$P = \tfrac{1}{2}\,|n|; \tag{18}$$

the minimum value of P' and $|P''|$ then depend on the parity of A. If A is odd, they are both $\frac{1}{2}$ and the ground state supermultiplet is

$$(\tfrac{1}{2}|n|,\ \tfrac{1}{2},\ \pm\tfrac{1}{2}) \qquad \text{(odd } A\text{)}. \tag{19}$$

To determine the sign of P'', we notice that according to (*10.14*–4, 8) and (18), we have

$$\begin{aligned} \Lambda_1 + \Lambda_2 &= \{N \text{ or } Z\} \\ \Lambda_1 - \Lambda_2 &= P' + P'' \geqslant 0, \end{aligned} \tag{20}$$

the notation $\{N \text{ or } Z\}$ representing the larger of the two numbers N, Z of neutrons or protons of the nucleus. According as this number $\{N \text{ or } Z\}$ is even or odd, the minimum value of $P' + P''$ is thus 0 or 1. Hence the rule: *for odd A, P'' is $+\frac{1}{2}$ or $-\frac{1}{2}$ according as $\{N$ or $Z\}$ is odd or even.* This rule may be expressed in other equivalent ways as follows:

10.311. *Sign of P'' for odd A* (21)

	$P=\frac{1}{2}\|n\|=$ $2t+\frac{1}{2}$	$P=\frac{1}{2}\|n\|=$ $2t-\frac{1}{2}$		N is even	N is odd
$A=4a+1$	$+$	$-$	$n>0$	$-$	$+$
$A=4a+3$	$-$	$+$	$n<0$	$+$	$-$

If A is even, the same argument immediately leads to the conclusion: *the minimum values of* (P', P'') *are* $(0, 0)$ *or* $(1, 0)$ *according as we have to do with an even or an odd nucleus* (*3.1*). This result cannot be valid, however, for odd nuclei with zero neutron excess, since it would imply a smaller value for P than for P'. In this exceptional case, we must therefore abandon (18) and (20); but since, according to (*10.14*–10), the minimum value of $P + P' + P''$ is then 1, we may take $(P, P', P'') = (1, 0, 0)$. Summarizing, we may state that the supermultiplet to which the ground state of an even mass nucleus belongs is fixed by the following rules:

	$n \neq 0$		$n = 0$
	$\lvert n \rvert = 4\nu$	$\lvert n \rvert = 4\nu + 2$	
$A = 4a$	$(\frac{1}{2}\lvert n \rvert, 0, 0)$	$(\frac{1}{2}\lvert n \rvert, 1, 0)$	$(0, 0, 0)$
$A = 4a + 2$	$(\frac{1}{2}\lvert n \rvert, 1, 0)$	$(\frac{1}{2}\lvert n \rvert, 0, 0)$	$(1, 0, 0)$

(22)

We can now draw a graph of Ξ' against of $P\ (= \frac{1}{2}\lvert n \rvert)$ by joining by straight lines the successive points of coordinates (P, Ξ'), the value of A being kept fixed. For odd A, such graphs will, according to (19), show a smooth increase with increasing P, while for even A, as shown by (22), this increase will be regularly interrupted by a succession of breaks. It is this characteristic difference which, so far as the approximative formula (9) may be trusted, would be responsible for the different behaviours of isobars of odd and even mass numbers (*3.2*). Likewise, it would account for the kinks occurring with a periodicity of 4 mass units in the mass-defect curve of the lightest stable nuclei (*3.221*).

10.32. *Mass-defects of light nuclei on Wigner's approximation.* To get the total binding energy of the ground state of a nucleus, we have to add to the nuclear potential energy term av $\mathcal{V}_W$, given by (9), the contribution from the kinetic and Coulomb energies. The latter can be written in the simple form (*2.21*–1), with (*2.1*–1); i.e., since $Z = \frac{1}{2} A - \mathrm{T}_3$,

$$\mathcal{G} = \mathcal{G}_0 \left[\tfrac{1}{2} A \left(\tfrac{1}{2} A - 1 \right) - \mathrm{T}_3 \left(A - \mathrm{T}_3 - 1 \right) \right]$$

$$\mathcal{G}_0 = \frac{3}{5} \cdot \frac{e^2}{r_0} \cdot \frac{1}{A^{\frac{1}{3}}} = \frac{0{,}6}{A^{\frac{1}{3}}}\ \mathrm{MeV}, \tag{23}$$

according to (*2.21*–6). For the kinetic energy, we shall adopt (for want of better) the estimate (*9.41*–7) based on the Fermi gas model approximation. This shows that the kinetic energy depends only on the mass number A, except for a relatively small correction term of the form kP^2/A; although this result concerns only even nuclei of large mass number, we may extend it to light nuclei, the value of the co-factor k being suitably specified for the different types of such nuclei. Collecting all terms, we get

for the total binding energy an expression of the type

$$|\mathcal{E}| = |\mathcal{E}_0(A)| - \Xi' \mathcal{L}(A) + \mathcal{G}_0(A - \mathrm{T}_3 - 1)\mathrm{T}_3 - k\frac{P^2}{A}; \qquad (24)$$

the coefficient $\mathcal{L}(A)$ of Ξ' is a positive, smoothly varying function of A; the function $\mathcal{E}_0(A)$ represents the sum of all the contributions depending only (or primarily) on A, which make up the greater part of the binding energy. Formula (24), which is due to WIGNER [*37b*], is not expected to hold for heavy nuclei owing to the perturbing effect of the Coulomb interactions on the supermultiplet system; for $A \lesssim 8$, the kinetic energy term becomes too uncertain and the formula becomes unreliable in this region too.

For intermediate nuclei (up to $A \approx 50$), Wigner's formula has been extensively compared with experiment by BARKAS [*39a*]. The observed mass-defects are first combined with the calculated Coulomb and kinetic energy corrections. The values so obtained for all nuclei with the same value of Ξ' yield a graph of $|\mathcal{E}_0(A)| - \Xi' \mathcal{L}(A)$ against A. Comparison of such graphs for different Ξ' gives the function $\mathcal{L}(A)$. Once this is known, $\mathcal{E}_0(A)$ can also be determined. The function $\mathcal{L}(A)$ derived in this way exhibits a smooth decrease with increasing A, starting from a value of about 3,8 mMU for $A = 8$, and it asymptotically tends to the form $50/A$ mMU (valid for $A \gtrsim 15$), in conformity with expectation (*10.31*).

As regards the function $|\mathcal{E}_0(A)|$, it is found to increase with A in a roughly linear manner, corresponding in the interval $A = 20 \dots 40$ to a value of the order of 9 mMU per nucleon. This increase is not altogether smooth, however: it exhibits two important breaks, one in the region $A = 16 \dots 20$, the other at $A = 40$. On the quasi-atomic model, such breaks can be interpreted as revealing the completion of orbital shells (*10.13*). If the Coulomb forces are sufficiently small, the stablest configurations of even mass nuclei are those of zero neutron excess; as long as this is the case (i.e. up to $A = 40$), orbital shells can be completely filled up with sets of two protons and two neutrons with saturated spins. The $1s$ shell would be completed with ^4He; then would begin the filling of a p shell, which should be called according to the usual nomenclature the $2p$ shell: this shell would be closed in ^{16}O. According to the evidence under discussion on the variation of mass-defect with mass number, the next shell would again be an s one, the $2s$ shell, terminating with ^{20}N. It would be followed by a d shell, the $3d$ shell, completely filled up in ^{40}Ca.

It must also be mentioned that Barkas was able to estimate the order of magnitude of the neglected variations of the spin dependent interaction within a group of isobaric nuclei and to correct the calculated binding energies for this omission. For this purpose, he compared isobars of mass $4a + 2$ with respective neutron excesses 0 and 2: the former is an odd nucleus, the latter an even one, the ground states of which, as shown by (22), both belong to the supermultiplet (1, 0, 0). In both, the two nucleons

in excess of a completely saturated configuration of $2a$ protons and $2a$ neutrons are symmetrically coupled as regards their space coordinates. But the nuclei will be expected to differ as to the spin multiplicity: the odd nucleus will be (*11.41*) in a triplet state, $S = 1$, while for the even one, $S = 0$. The effective spin dependent interactions will therefore be of the ^3S and ^1S types, respectively. Their difference can readily be derived from that between the empirical mass-defects, when account is further taken of the easily estimated difference in Coulomb energy. In this way, values are found for the spin energy difference ranging from 4,6 mMU for the pair ^6He — ^6Li to 0,6 mMU for the pair ^{30}Si — ^{30}P. The sign is such that the ^3S force is a stronger attraction than the ^1S attractive force. The correction adopted by Barkas consists in adding half the difference to the binding energy $\mathcal{E}_0(A)$ of the odd isobar and subtracting the same amount from that of the even one. The effect of the spin dependent interactions on the binding energies of the ground state thus appears on the whole to be rather small; this is somewhat surprising in view of the fact that the proportion of these forces in the nuclear potential should by no means be small compared with the ordinary and Majorana forces. Also in the interpretation of another important group of empirical data, pertaining to β-active nuclei (*A1.2*), Wigner's approximation meets with a larger amount of success than one might expect on that account.

10.33. *Rough estimate of spin dependent interactions.* The contribution of the spin dependent forces to the expectation value of the potential energy in a supermultiplet state can easily be estimated in the crude approximation afforded by the long range fiction (*10.31*). For if we replace the potential function $\mathcal{J}(r)$ by a constant $\mathcal{J}$, we have just to compute the expectation values

$$\text{av}\, \tfrac{1}{2} \sum_{i,k} P_\sigma^{(ik)} \quad , \quad \text{av}\, \tfrac{1}{2} \sum_{i,k} P_\tau^{(ik)}$$

in a state with given spin and isotopic quantum numbers S, T. Following a well-known procedure due to Dirac, one starts from

$$\text{av} \sum_{i,k} \vec{\sigma}^{(i)} \vec{\sigma}^{(k)} = \text{av}\, [(\sum_i \vec{\sigma}^{(i)})^2 - \sum_i (\vec{\sigma}^{(i)})^2] = 4\, S\,(S+1) - 3A, \tag{25}$$

whence

$$\text{av}\, \tfrac{1}{2} \sum_{i,k} P_\sigma^{(ik)} = \text{av}\, \tfrac{1}{2} \sum_{i,k} \frac{1 + \vec{\sigma}^{(i)} \vec{\sigma}^{(k)}}{2} = \tfrac{1}{4} A\,(A-4) + S\,(S+1). \tag{26}$$

Similarly,

$$\text{av}\, \tfrac{1}{2} \sum_{i,k} P_\tau^{(ik)} = \tfrac{1}{4} A\,(A-4) + \mathrm{T}\,(\mathrm{T}+1). \tag{27}$$

Combining these results with formula (10), we find for the expectation value of the general interaction operator $\mathcal{V} = \mathcal{J}\,\mathcal{O}$, with $\mathcal{O}$ given by (*8.2–4*),

$$\begin{aligned} \text{av}\, \mathcal{V}_{\text{long range}} = \mathcal{J}\,\{ & \tfrac{1}{2} A\,(A-1) a_0 + [S(S+1) - \tfrac{3}{4} A]\, 2\, a_\sigma + [\mathrm{T}(\mathrm{T}+1) - \tfrac{3}{4} A] 2 a_\tau \\ & + [P(P+4) + P'(P'+2) + P''^2 - S(S+1) - \mathrm{T}(\mathrm{T}+1) - \tfrac{9}{4} A]\, 2\, a_{\sigma\tau} \}. \end{aligned} \tag{28}$$

In deriving this expression, use has also been made of the formulae (*8.2–5*) and (15).

10.34. *Coulomb energy of light nuclei.* It has been noted in *3.3* that a marked influence of exchange Coulomb interaction on the mass-defect can be perceived for the lightest nuclei. FEENBERG and GOERTZEL [46a] have recently analyzed this effect on more general assumptions; in particular, their discussion does not involve the Slater-determinant representation of the wave-function, on which Phillips and Feenberg's just recalled interpretation in terms of exchange Coulomb energy was based. The more fundamental distinction, which replaces that between ordinary and exchange interactions, is that between symmetrically and antisymmetrically coupled pairs (*10.31*) of protons. Such pairs are characterized by the operators

$$\begin{aligned} \Sigma_p^{(ik)} &= \tfrac{1}{2}\,(1 + P_x^{(ik)})\,\mathsf{T}_-^{(i)}\,\mathsf{T}_-^{(k)} \\ \mathsf{A}_p^{(ik)} &= \tfrac{1}{2}\,(1 - P_x^{(ik)})\,\mathsf{T}_-^{(i)}\,\mathsf{T}_-^{(k)}, \end{aligned} \tag{29}$$

the expectation values of which we shall denote by p_s, p_a:

$$p_s = \mathrm{av}\ \Sigma_p^{(12)} \quad , \quad p_a = \mathrm{av}\ \mathsf{A}_p^{(12)}; \tag{30}$$

these quantities are, of course, the same for any pair (ik), and $\frac{1}{2}A(A-1)\,p_s$, $\frac{1}{2}A(A-1)\,p_a$ represent the respective numbers of proton pairs for which the symmetrized normal form of the spatial wave-functions is symmetrical, and the antrisymmetrized normal form antisymmetrical.

The average contributions of the two kinds of proton pairs to the expectation value of the Coulomb energy will be denoted by

$$\mathcal{G}_s = \frac{1}{p_s}\ \mathrm{av}\ \frac{e^2}{r^{(12)}}\,\Sigma_p^{(12)} \quad , \quad \mathcal{G}_a = \frac{1}{p_a}\ \mathrm{av}\ \frac{e^2}{r^{(12)}}\,\mathsf{A}_p^{(12)}. \tag{31}$$

The total Coulomb energy,

$$\begin{aligned} \mathcal{G} = \mathrm{av}\ \tfrac{1}{2}\sum_{i,k}\frac{e^2}{r^{(ik)}}\,\mathsf{T}_-^{(i)}\mathsf{T}_-^{(k)} &= \tfrac{1}{2}\,A\,(A-1)\ \mathrm{av}\ \frac{e^2}{r^{(12)}}\,\mathsf{T}_-^{(1)}\mathsf{T}_-^{(2)} \\ &= \tfrac{1}{2}\,A\,(A-1)\,(p_s\,\mathcal{G}_s + p_a\,\mathcal{G}_a), \end{aligned} \tag{32}$$

can be decomposed into

$$\mathcal{G} = \tfrac{1}{2}\,A\,(A-1)\,[(p_s + p_a)\,\mathcal{G}^+ + (\tfrac{3}{4}\,p_s - \tfrac{1}{4}\,p_a)\,\mathcal{G}^-)], \tag{33}$$

with

$$\begin{aligned} \mathcal{G}^+ &= \tfrac{1}{4}\,\mathcal{G}_s + \tfrac{3}{4}\,\mathcal{G}_a \\ \mathcal{G}^- &= \mathcal{G}_s - \mathcal{G}_a. \end{aligned} \tag{34}$$

The meaning of this decomposition will become clear if the limiting case of heavy nuclei is considered: in this case, $\mathcal{G}_s$ and $\mathcal{G}_a$ tend to become equal (their common value being $\mathcal{G}_0$), while (as we shall see presently) the

limiting value of the ratio p_s/p_a is $\frac{1}{3}$. This last circumstance justifies the choice of the combination G^+ to represent the main part of G; the term in G^- is then a correction, only important for light nuclei and exhibiting, as we shall show, a strong dependence on the parity of Z.

In order to determine p_s and p_a, we have in the first place the obvious relation

$$\tfrac{1}{2} A(A-1)(p_s + p_a) = \tfrac{1}{2} Z(Z-1). \tag{35}$$

The difference $p_s - p_a$ can next be found from

$$\begin{aligned} \tfrac{1}{2} A(A-1)(p_s - p_a) &= \operatorname{av} \tfrac{1}{2} \sum_{i,k} P_x^{(ik)} T_-^{(i)} T_-^{(k)} \\ &= -\operatorname{av} \tfrac{1}{2} \sum_{i,k} P_\sigma^{(ik)} T_-^{(i)} T_-^{(k)}, \end{aligned} \tag{36}$$

in virtue of $P_x = -P_\sigma P_\tau$ and

$$P_\tau^{(ik)} T_-^{(i)} T_-^{(k)} = T_-^{(i)} T_-^{(k)}. \tag{37}$$

In exactly the same way as (*10.33*–26), we derive

$$\operatorname{av} \tfrac{1}{2} \sum_{i,k} P_\sigma^{(ik)} T_-^{(i)} T_-^{(k)} = \tfrac{1}{4} Z(Z-4) + S_p(S_p + 1), \tag{38}$$

S_p being the quantum number of the total proton spin, i.e. 0 for even Z and $\frac{1}{2}$ for odd Z, or

$$S_p = \tfrac{1}{4}[1 - (-1)^Z]. \tag{39}$$

Inserting (38) and (39) into (36), we get

$$\tfrac{1}{2} A(A-1)(p_s - p_a) = -\tfrac{1}{4} Z(Z-4) + \tfrac{3}{8}[1 - (-1)^Z]. \tag{40}$$

A first consequence of (35) and (40) is the property already mentioned: $\lim_{Z\to\infty} (p_s/p_a) = \frac{1}{3}$. More generally, the coefficients of formula (33) can be expressed entirely in terms of Z:

$$G = \tfrac{1}{2} Z(Z-1)\, G^+ + \tfrac{3}{2}[Z - \tfrac{1}{2} + \tfrac{1}{2}(-1)^Z]\, G^-. \tag{41}$$

If we now assume (just as we did for the functions $\mathcal{L}^\pm$ in *10.31*) that G^+ and G^- are smoothly varying functions of A, we see that the coefficient of G^- will give rise to fluctuations of the value of G according to the parity of Z, in qualitative agreement with the empirical data. The data on differences of Coulomb energy, obtained from isobaric pairs, would in fact allow us to determine empirically the variation of G^+ and G^- with A; an element of uncertainty lies in the other possible cause of fluctuations in the value of Coulomb energies, viz. the variation of nuclear dimensions emphasized by Bethe (*3.3*). The detailed discussion, carried out by Feenberg and Goertzel, shows that no appreciable difference between G_s and G_a subsists for mass numbers larger than 17, and that the effect of this difference is considerable only for mass numbers up to 11.

CHAPTER XI

SATURATION PROPERTIES OF NUCLEAR FORCES

11.0. In this Chapter, we shall try to formulate necessary and sufficient conditions to be fulfilled by the nuclear potential (assumed to be a *static central* one of the same sign at all distances) in order that it should give rise to the saturation properties observed in heavy nuclei. For this purpose, the average interaction energy in a specified stationary state of such a nucleus must be set up, and this will necessarily involve the use of some approximate model. Since the conditions in question take the form of inequalities to be satisfied by the interaction parameters occurring in the general expression (*8.2*–1), a crude approximation such as that afforded by models of the individual type will do. It will turn out that it even suffices to fix the values of these parameters with considerable precision.

11.1. The saturation requirements

11.11. *Decomposition of mean nuclear energy into ordinary and exchange terms.* In any individual nuclear model, the eigenfunctions of the stationary states are to be built up of individual nucleon wave-functions of the form *

$$\psi_n(Q) = \varphi_n(\vec{x})\, v_\pm(\sigma_z')\, u_\pm(\tau_3'), \tag{1}$$

the v's and u's being defined as in *4.32, 4.331*. In order to satisfy the exclusion principle, we may write them as linear combinations of Slater-determinants

$$\frac{1}{\sqrt{A!}} \begin{vmatrix} \psi_{n_1}(Q^{(1)}) \dots \psi_{n_1}(Q^{(A)}) \\ \dots\dots\dots\dots \\ \psi_{n_A}(Q^{(1)}) \dots \psi_{n_A}(Q^{(A)}) \end{vmatrix}, \tag{2}$$

each of which corresponds to a configuration of the kind represented by fig. *10.14*. The nuclear interaction potential being of the general form (*8.2*–1) and, the Coulomb potential disregarded, any stationary state will belong to definite eigenvalues of the 3-component T_3 of the total isotopic vector (*10.12*–2) and of the z-component S_z of the total spin (*10.14*–6). But in general, it cannot be ascribed an eigenvalue of the quantity Y_z (*10.14*–7), since the latter commutes only with the $P_\sigma P_\tau$'s, but not generally with the P_σ's and P_τ's separately. Hence it is necessary to combine several Slater-determinants, since any such determinant is an

* We do not consider any dependence on ϱ_3', since we work in the non-relativistic approximation.

eigenfunction of Y_z also. Still, there are stationary states in which Y_z also is well-defined: they belong to a common eigenvalue 0 of all commutators $[P_\sigma, \mathsf{Y}_z]$, $[P_\tau, \mathsf{Y}_z]$. This is obviously the case, e.g., if all $\tau_3^{(i)'}$ are the same, i.e. if we consider a "nucleus" consisting solely of protons or of neutrons; again, if all $\sigma_z^{(i)'}$ are the same, corresponding to a nuclear state with an extreme value $\pm \frac{1}{2}A$ of the z-component of the total spin; more generally, if the configuration is such that — on account of the exclusion principle — no neutron-proton pair with opposite spins can be found which would allow an interchange (as an example of such a configuration, we may mention any one with saturated spins, as that of the ground state; a more general case is that represented by fig. *10.14*; an instance of configurations *not* satisfying this condition is met with in *11.4* below). The eigenfunctions of such stationary states are single Slater-determinants of the type (2). Except in the last mentioned instance, we shall here exclusively have to do with states of this special class.

For the state represented by the Slater-determinant (2), the expectation value of the nuclear interaction energy

$$\mathcal{V}_{\text{nucl}} = \tfrac{1}{2} \sum_{i,k} \mathcal{V}_{\text{nucl}}^{(ik)} \tag{3}$$

is easily reduced to the following form

$$\begin{aligned} \overline{\mathcal{V}} &= \text{av}\ \mathcal{V}_{\text{nucl}} \\ &= \tfrac{1}{2} \sum_{n_1, n_2} \int \psi_{n_1}^*(Q^{(1)})\, \psi_{n_2}^*(Q^{(2)}) \mathcal{V}_{\text{nucl}}^{(12)} \left[\psi_{n_1}(Q^{(1)})\, \psi_{n_2}(Q^{(2)}) - \psi_{n_2}(Q^{(1)})\, \psi_{n_1}(Q^{(2)})\right]; \end{aligned} \tag{4}$$

in this formula, the integral sign indicates summation over the spin and isotopic variables in addition to integration over space, while the summations over n_1 and n_2 independently extend over all the occupied individual nucleon states ψ_n. Owing to the special form (*8.2*–1) of $\mathcal{V}_{\text{nucl}}^{(12)}$, we can perform explicitly the summations over spin and isotopic variables and so express $\overline{\mathcal{V}}$ as a sum of space integrals of the ordinary and exchange types. In fact, we observe that, according to (*8.2*–2),

$$\sum_{\tau_3^{(1)'}, \tau_3^{(2)'}} u_\pm(1)\, u_\pm(2)\, \boldsymbol{\tau}^{(1)} \boldsymbol{\tau}^{(2)}\, u_\pm(1)\, u_\pm(2) = \pm 1, \tag{5}$$

the sign being $+$ or $-$ according as the sign indices of the u's are the same or opposite for the two particles, and that

$$\sum_{\tau_3^{(1)'}, \tau_3^{(2)'}} u_+(1)\, u_-(2)\, \boldsymbol{\tau}^{(1)} \boldsymbol{\tau}^{(2)}\, u_-(1)\, u_+(2) = 2; \tag{6}$$

similar formulae hold for $\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}$.

Beginning with the τ-summation, we get the separation of $\overline{\mathcal{V}}$ into neutron-neutron, proton-proton and proton-neutron energies:

$$\overline{\mathcal{V}} = \sum_{m_t} \overline{\mathcal{V}}_{m_t} = \overline{\mathcal{V}}_{+1} + \overline{\mathcal{V}}_{-1} + \overline{\mathcal{V}}_0. \tag{7}$$

One finds

$$\begin{aligned}\overline{\mathcal{V}}_{\pm 1} &= \tfrac{1}{2} \sum_{n_1}^{(\pm)} \sum_{n_2}^{(\pm)} \int \chi^*_{n_1}(1)\, \chi^*_{n_2}(2)\, \mathcal{V}_{\pm 1}\, [\chi_{n_1}(1)\, \chi_{n_2}(2) - \chi_{n_2}(1)\, \chi_{n_1}(2)] \\ \overline{\mathcal{V}}_0 &= \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi^*_{n_1}(1)\, \chi^*_{n_2}(2)\, \mathcal{V}_0^{\mathrm{ord}}\, \chi_{n_1}(1)\, \chi_{n_2}(2) \\ &\quad - \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi^*_{n_1}(1)\, \chi^*_{n_2}(2)\, \mathcal{V}_0^{\mathrm{exch}}\, \chi_{n_2}(1)\, \chi_{n_1}(2)\,;\end{aligned} \tag{8}$$

in these formulae, one has put

$$\chi_n = \varphi_n(\vec{x})\, v_{\pm}(\sigma'_z) \tag{9}$$

and the symbols $\Sigma^{(+)}$, $\Sigma^{(-)}$ represent summations extending over the occupied neutron states and occupied proton states, respectively. Further, the effective potential operators occurring in (8) are defined as follows:

$$\begin{aligned}\mathcal{V}_{\pm 1} &= [a_0 + a_\tau + (a_\sigma + a_{\sigma\tau})\, \vec{\sigma}^{(1)}\, \vec{\sigma}^{(2)}]\, J(r) \\ \mathcal{V}_0^{\mathrm{ord}} &= [a_0 - a_\tau + (a_\sigma - a_{\sigma\tau})\, \vec{\sigma}^{(1)}\, \vec{\sigma}^{(2)}]\, J(r) \\ \mathcal{V}_0^{\mathrm{exch}} &= 2\, [a_\tau + a_{\sigma\tau}\, \vec{\sigma}^{(1)}\, \vec{\sigma}^{(2)}]\, J(r).\end{aligned} \tag{10}$$

We see that both the like-nucleon energy $\mathcal{V}_{+1}$ and the proton-neutron energy $\overline{\mathcal{V}}_0$ consist of two parts of the ordinary and exchange types, respectively; moreover, in the proton-neutron case, the effective potentials of ordinary and exchange types are themselves different.

The spin-summation has now to be carried out in each of these various terms. By way of illustration, let us treat the exchange terms of $\overline{\mathcal{V}}_0$, which we denote by

$$\overline{\mathcal{V}}_0^{\mathrm{exch}} = \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi^*_{n_1}(1)\, \chi^*_{n_2}(2)\, \mathcal{V}_0^{\mathrm{exch}}\, \chi_{n_2}(1)\, \chi_{n_1}(2). \tag{11}$$

The summations $\Sigma^{(+)}$ must further be analysed into summations over all occupied (proton or neutron) states of the one or the other spin orientation, which we shall denote by $\uparrow\Sigma^{(\pm)}$ and $\downarrow\Sigma^{(\pm)}$. We may then group together the terms for which the spin orientations in the two independent summations $\sum\limits_{n_1}^{(+)} \sum\limits_{n_2}^{(-)}$ are the same, and those for which they are opposite. According to the formulae analogous to (5) and (6), the terms of the former group contain the factor $2(a_\tau + a_{\sigma\tau})$, those of the latter the factor $4a_{\sigma\tau}$. In order to write down the remaining space integrals, it is convenient to introduce quantities, called *mixed densities*, of the following type:

$$\uparrow\varrho_+ (\vec{x}^{(1)}, \vec{x}^{(2)}) = \uparrow\sum_n^{(+)} \varphi^*_n (\vec{x}^{(1)})\, \varphi_n (\vec{x}^{(2)}), \tag{12}$$

and similarly $\downarrow\varrho_+$, $\uparrow\varrho_-$, $\downarrow\varrho_-$; the summation in (12), e.g., extends over all occupied neutron states of the specified spin orientation. If we put

$\vec{x}^{(1)} = \vec{x}^{(2)} = \vec{x}$, we get the corresponding ordinary densities, which we may denote by the same symbols, but with a single argument $\vec{x}$, e.g.

$$^{\uparrow}\varrho_{+}(\vec{x}) \equiv {}^{\uparrow}\varrho_{+}(\vec{x}, \vec{x}) = {}^{\uparrow}\sum_{n}^{(+)} \varphi_n^{*}(\vec{x})\, \varphi_n(\vec{x}) \,. \tag{13}$$

With this notation, the (exchange) integrals occurring in $\overline{\mathcal{V}}_0^{\text{exch}}$ may be symbolized in the following way:

$$\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)_{\text{exch}} \equiv \Re \int {}^{\uparrow}\varrho_{+}(\vec{x}^{(1)}, \vec{x}^{(2)})\; J(|\vec{x}^{(1)} - \vec{x}^{(2)}|)\; {}^{\downarrow}\varrho_{-}(\vec{x}^{(2)}, \vec{x}^{(1)})\; dv^{(1)}\, dv^{(2)}, \tag{14}$$

and the result takes the form

$$\begin{aligned}
\overline{\mathcal{V}}_0^{\text{exch}} &= {}^{\pi}\overline{\mathcal{V}}_0^{\text{exch}} + {}^{\alpha}\overline{\mathcal{V}}_0^{\text{exch}} \\
{}^{\pi}\overline{\mathcal{V}}_0^{\text{exch}} &= 2\,(a_\tau + a_{\sigma\tau})\left[\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\-\end{smallmatrix}\right) + \left(\begin{smallmatrix}\downarrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)\right] \\
{}^{\alpha}\overline{\mathcal{V}}_0^{\text{exch}} &= \qquad 4\,a_{\sigma\tau}\left[\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right) + \left(\begin{smallmatrix}\downarrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\-\end{smallmatrix}\right)\right].
\end{aligned} \tag{15}$$

If we had considered the ordinary term $\overline{\mathcal{V}}_0^{\text{ord}}$, e.g., the same procedure would have led to a expression of exactly the same type:

$$\overline{\mathcal{V}}_0^{\text{ord}} = {}^{\pi}\overline{\mathcal{V}}_0^{\text{ord}} + {}^{\alpha}\overline{\mathcal{V}}_0^{\text{ord}}, \tag{16}$$

but the co-factors in ${}^{\pi}\overline{\mathcal{V}}_0^{\text{ord}}$ and ${}^{\alpha}\overline{\mathcal{V}}_0^{\text{ord}}$ would have been different combinations of the a's, and the symbols such as $\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)$ would have had another meaning, viz.

$$\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)_{\text{ord}} = \int {}^{\uparrow}\varrho_{+}(\vec{x}^{(1)})\, J(|\vec{x}^{(1)} - \vec{x}^{(2)}|)\, {}^{\downarrow}\varrho_{-}(\vec{x}^{(2)})\, dv^{(1)}\, dv^{(2)} \,. \tag{17}$$

Quite generally, therefore, we get the expression

$$\mathcal{V}_{m_t} = {}^{\pi}\overline{\mathcal{V}}_{m_t}^{\text{ord}} + {}^{\alpha}\overline{\mathcal{V}}_{m_t}^{\text{ord}} - {}^{\pi}\overline{\mathcal{V}}_{m_t}^{\text{exch}} - {}^{\alpha}\overline{\mathcal{V}}_{m_t}^{\text{exch}} \,, \tag{18}$$

in which each term appears as a combination of integrals of the form (14) or (17), with co-factors depending on the a's. The various combinations of integrals and co-factors are summarized in the following table; the same combination of integrals has to be taken for the ordinary and exchange terms of the same type, the meaning of the symbols being given by (14) or (17) as the case may be:

11.11—1. *Decomposition of mean nuclear energy*

Type of term	Combination of integrals	Co-factor: Ordinary term	Co-factor: Exchange term
${}^{\pi}\overline{\mathcal{V}}_{\pm 1}$	$\frac{1}{2}\left[\left(\begin{smallmatrix}\uparrow\\\pm\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\\pm\end{smallmatrix}\right) + \left(\begin{smallmatrix}\downarrow\\\pm\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\\pm\end{smallmatrix}\right)\right]$	$a_0 + a_\tau + a_\sigma + a_{\sigma\tau}$	$a_0 + a_\tau + a_\sigma + a_{\sigma\tau}$
${}^{\alpha}\overline{\mathcal{V}}_{\pm 1}$	$\left(\begin{smallmatrix}\uparrow\\\pm\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\\pm\end{smallmatrix}\right)$	$a_0 + a_\tau - a_\sigma - a_{\sigma\tau}$	$2\,(a_\sigma + a_{\sigma\tau})$
${}^{\pi}\overline{\mathcal{V}}_0$	$\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\-\end{smallmatrix}\right) + \left(\begin{smallmatrix}\downarrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)$	$a_0 - a_\tau + a_\sigma - a_{\sigma\tau}$	$2\,(a_\tau + a_{\sigma\tau})$
${}^{\alpha}\overline{\mathcal{V}}_0$	$\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right) + \left(\begin{smallmatrix}\downarrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\-\end{smallmatrix}\right)$	$a_0 - a_\tau - a_\sigma + a_{\sigma\tau}$	$4a_{\sigma\tau}$

A considerable simplification occurs for configurations in which all the nucleon spins are saturated. In this case,

$$^{\uparrow}\varrho_{\pm} = {}^{\downarrow}\varrho_{\pm} = \tfrac{1}{2}\,\varrho_{\pm}\,, \tag{19}$$

$\varrho_{\pm}$ denoting the total (ordinary or mixed) density of the neutrons or protons; we are thus left with 4 different types of integrals only, which may be denoted by symbols such as

$$(+|-) = \begin{cases} \int \varrho_{+}(\vec{x}^{(1)})\, J\,(r^{(12)})\, \varrho_{-}(\vec{x}^{(2)})\, dv^{(1)}\, dv^{(2)} & \text{(ordinary)} \\ \Re \int \varrho_{+}(\vec{x}^{(1)}, \vec{x}^{(2)})\, J\,(r^{(12)})\, \varrho_{-}(\vec{x}^{(2)}, \vec{x}^{(1)})\, dv^{(1)}\, dv^{(2)} & \text{(exchange).} \end{cases} \tag{20}$$

Instead of the expression (18), we have, more simply,

$$\mathcal{V}_{m_t} = \overline{\mathcal{V}}^{\text{ord}}_{m_t} - \overline{\mathcal{V}}^{\text{exch}}_{m_t}\,, \tag{21}$$

with the following integrals and co-factors:

11.11–2. *Decomposition of mean nuclear energy in states with saturated spins*

Type of term	Integrals	Co-factor	
		Ordinary	Exchange
$\overline{\mathcal{V}}_{\pm 1}$	$\frac{1}{4}(+\,|\,+)$ or $\frac{1}{4}(-\,|\,-)$	$2\,(a_0 + a_\tau)$	$a_0 + a_\tau + 3a_\sigma + 3a_{\sigma\tau} = 2\,(a_0 + a_\tau) + q$
$\mathcal{V}_0$	$(+\,|\,-)$	$a_0 - a_\tau$	$a_\tau + 3a_{\sigma\tau} = a_0 - a_\tau + \frac{1}{4}\,(q+3)$

The equivalent expressions for the coefficients of the exchange integrals follow from (*8.2*–12).

In particular, consider the ground state of a nucleus containing as many neutrons as protons; there is then just one ordinary and one exchange integral

$$\begin{aligned} I &= \int \varrho\,(\vec{x}^{(1)})\, J\,(r^{(12)})\, \varrho\,(\vec{x}^{(2)})\, dv^{(1)}\, dv^{(2)} \\ X &= \int \varrho\,(\vec{x}^{(1)}, \vec{x}^{(2)})\, J\,(r^{(12)})\;\; \varrho\,(\vec{x}^{(2)}, \vec{x}^{(1)})\, dv^{(1)}\, dv^{(2)}\,, \end{aligned} \tag{22}$$

with

$$\varrho\,(\vec{x}^{(1)}, \vec{x}^{(2)}) = \sum_n \varphi_n^*\,(\vec{x}^{(1)})\, \varphi_n(\vec{x}^{(2)})\,, \tag{23}$$

the summation extending over the occupied (lowest) states. The mean nuclear energy is then, by (7) and (21), since $\varrho_+ = \varrho_- = \varrho$,

$$\begin{aligned} \overline{\mathcal{V}} &= 2a_0\, I - \tfrac{1}{2}\,(a_0 + 3a_\tau + 3a_\sigma + 9a_{\sigma\tau})\, X \\ &= 2a_0\, I - [2a_0 + \tfrac{3}{4}\,(1 + q)]\, X\,; \end{aligned} \tag{24}$$

it depends only on a_0 (and q).

11.12. *Saturation properties of exchange interaction terms.* The asymptotic behaviours for large numbers of nucleons of the ordinary and exchange integrals are radically different. The former, depending on a product of two densities, vary, roughly, as the product of the two proton or neutron numbers, i.e. quadratically with A and consequently do not exhibit any saturation. The mixed densities, however, which enter into the definition of the exchange integrals, reduce asymptotically to a delta function $\delta(\vec{x}^{(1)} - \vec{x}^{(2)})$; indeed, for a very heavy nucleus, the summation in (12) may approximately be extended over *all* states φ_n, which by the completeness relation just yields the delta function. In general, the number of neutrons being larger than the number of protons, the mixed neutron densities will be nearer their asymptotic behaviour than the mixed proton densities. If, then, we replace in (14) the mixed neutron density ${}^{\uparrow}\varrho_{+}$ by the δ-function, we get

$$\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\downarrow\\-\end{smallmatrix}\right)_{\text{exch}} \simeq J(0) \int {}^{\downarrow}\varrho_{-}(\vec{x})\, dv\,, \tag{25}$$

a quantity approximately proportional to the number of protons, i.e. linear in A, as required to account for the saturation of nuclear bonds. This conclusion is clearly valid for all exchange integrals.

According to this argument, a Wigner potential in the sense of *4.341* will give rise to an ordinary non-saturation interaction and to an exchange interaction exhibiting saturation: this is the well-known situation in the theory of chemical binding. If the potential is of the Majorana type, we have just the reverse situation: the ordinary interaction terms possess the saturation property.

11.121. *Velocity dependent potentials.* Another extensive class of energy operators with saturation properties has been pointed out by WHEELER [36] *. In general, an effective potential operator may depend both on the relative coordinates $\vec{x}$ and on the conjugate relative momentum operators $\vec{p} \equiv \frac{\hbar}{i}$ grad of the nucleon pair. It can then also be expressed as an integral operator, as follows:

$$\mathcal{V}_{\text{eff}}\left(\vec{x}, \frac{\hbar}{i}\,\text{grad}\right) \psi(\vec{x}) = \int J(\vec{x}, \vec{\xi})\, \psi(\xi)\, dv_{\xi}\,, \tag{26}$$

with

$$J(\vec{x}, \vec{\xi}) = (2\pi\hbar)^{-3} \int \mathcal{V}_{\text{eff}}(\vec{x}, \vec{p})\, e^{\frac{i}{\hbar}(\vec{x}-\vec{\xi})\vec{p}}\, dv_p\,. \tag{27}$$

The equivalence of the two representations is easily verified by introducing in (27) the Taylor expansion of $\mathcal{V}_{\text{eff}}$ with respect to $\vec{p}$, substituting the result in the right-hand side of (26), transforming this by partial integrations with respect to $\vec{\xi}$, and using the Fourier integral theorem, or the equivalent symbolic formula (*N.13*) for $\delta(\vec{x} - \vec{\xi})$.

* See also WAY and WHEELER [36], WAY [37].

Now, the ordinary integrals for such a potential are, according to (17), (13), in which we go over to relative coordinates by means of (*4.311–2, 3*), of the general form

$$\left(\begin{smallmatrix}\uparrow & \downarrow \\ + & -\end{smallmatrix}\right)_{\rm ord} = \Re\ {}^{\uparrow}\sum_{n_1}^{(+)}\ {}^{\downarrow}\sum_{n_2}^{(-)} \int \varphi^*_{n_1}(\vec{X}+\tfrac{1}{2}\vec{x})\, \varphi^*_{n_2}(\vec{X}-\tfrac{1}{2}\vec{x})\, J(\vec{x},\vec{\xi})\cdot$$

$$\varphi_{n_1}(\vec{X}+\tfrac{1}{2}\vec{\xi})\,\varphi_{n_2}(\vec{X}-\tfrac{1}{2}\vec{\xi})\, dv_X\, dv_x\, dv_\xi \qquad (28)$$

$$= \Re \int {}^{\uparrow}\varrho_+(\vec{X}+\tfrac{1}{2}\vec{x}, \vec{X}+\tfrac{1}{2}\vec{\xi})\, J(\vec{x},\vec{\xi})\, {}^{\downarrow}\varrho_-\,(\vec{X}-\tfrac{1}{2}\vec{x}, \vec{X}-\tfrac{1}{2}\vec{\xi})\, dv_X\, dv_x\, dv_\xi\,.$$

They thus depend on *mixed* densities. In absolute value, a mixed density such as (12) presents a maximum at $\vec{x}^{(1)} = \vec{x}^{(2)}$, which becomes higher and narrower when more states are occupied. The behaviour of the integral (28) is essentially different according as $J(\vec{x},\vec{\xi})$ varies more or less rapidly than the narrower mixed density factor when $\vec{x}-\vec{\xi}$ varies. In the first case, we get no saturation: an extreme example is that of an ordinary Wigner potential $J(\vec{x},\vec{\xi}) = J(\vec{x})\,\delta(\vec{x}-\vec{\xi})$. But if the variation of $J(\vec{x},\vec{\xi})$ is sufficiently slow over the domain in which the narrower mixed density is appreciable, we may (as above) replace that mixed density (${}^{\uparrow}\varrho_+$, say) by $\delta\left(\frac{\vec{x}-\vec{\xi}}{2}\right) = 8\,\delta(\vec{x}-\vec{\xi})$, and (28) becomes

$$\left(\begin{smallmatrix}\uparrow & \downarrow \\ + & -\end{smallmatrix}\right)_{\rm ord} \sim \int 8\, J(\vec{x},\vec{x})\ {}^{\downarrow}\varrho_-(\vec{X}-\tfrac{1}{2}\vec{x})\, dv_X\, dv_x\,. \qquad (29)$$

On the Fermi gas model, the density being approximately constant over the whole nucleus, (29) may be written

$$\left(\begin{smallmatrix}\uparrow & \downarrow \\ + & -\end{smallmatrix}\right)_{\rm ord} \simeq \int 8\, J(\vec{x},\vec{x})\, dv_x \cdot \int {}^{\downarrow}\varrho_-(\vec{X})\, dv_X\,, \qquad (30)$$

which exhibits the saturation property.

The condition of slow variation of $J(\vec{x},\vec{\xi})$ means that we have to do with an essential *velocity dependence* of the effective potential. In fact, according to (27), the function $\mathcal{V}_{\rm eff}(\vec{x},\vec{p})$ must then have appreciable values for all $\vec{p}$ comprised within a domain $|\vec{p}| < p_0$, if $\hbar/p_0$ represents the dimensions of the region in which the mixed density is $\neq 0$; on the Fermi gas model, this limiting momentum is of the order of the maximum momentum p_m (*9.41–3*). We shall not further discuss this class of nuclear potentials giving rise to saturation effects, referring the reader to the papers quoted above.

It may be observed that the Majorana potential might be regarded as belonging to the type (26), (27), viz. if

$$\mathcal{V}_{\rm eff}(\vec{x},\vec{p}) = J(\vec{x})\, e^{-2i\vec{x}\vec{p}/\hbar} \qquad (31)$$

or

$$\mathcal{J}(\vec{x}, \vec{\xi}) = \mathcal{J}(\vec{x})\,\delta(\vec{x} + \vec{\xi}). \tag{32}$$

Although (30) also formally holds in this case, — actually reducing to (25), — we have here to do with a rather singular case, in which the velocity dependence embodied in (31) is only apparent, on account of the exclusion principle (*4.341*–32).

11.13. *The saturation requirements.* The total energy of a stationary state is made up of three contributions: kinetic energy, nuclear energy and Coulomb energy:

$$\mathcal{E} = \mathcal{K} + \overline{\mathcal{V}} + \mathcal{G}. \tag{33}$$

Since $\mathcal{K}$ and $\mathcal{G}$ are positive *, a state of binding ($\mathcal{E} < 0$) necessitates an attractive nuclear interaction stronger in absolute value than the sum of kinetic and Coulomb energy. The absolute value of the binding energy is given by

$$|\mathcal{E}| = |\overline{\mathcal{V}}| - (\mathcal{K} + \mathcal{G}). \tag{34}$$

The expression (*9.41*–7) for the kinetic energy practically satisfies the condition of being proportional to A, so that we may confine the discussion to the potential energy $\overline{\mathcal{V}} + \mathcal{G}$. This energy is composed of terms linear in A and of terms approximately quadratic in A: these will be called the saturation and non-saturation terms, respectively.

In the first place, we have to require that the non-saturation part of the potential energy should in no case give rise to an attraction, since for heavy nuclei this attraction would become preponderant and give rise to stable configurations not exhibiting the saturation property. The Coulomb energy is always repulsive, but numerically so insignificant that it can safely be left out of consideration. The *first saturation requirement* has therefore to be formulated for the non-saturation part of the nuclear energy $\overline{\mathcal{V}}$.

In the second place, we have to take care that, for stable states of actual nuclei, the saturation part of the nuclear energy yields an attraction, exceeding in absolute value the total positive contribution to the energy, arising from the non-saturation part of the potential energy and from the kinetic energy. This will be our *second saturation requirement.*

11.2. The first saturation requirement

11.21. *Necessary saturation conditions.* Let us consider the mean nuclear energy of some specified stationary state. In order to satisfy the first saturation requirement, we have to express that the non-saturation

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* Strictly speaking, the Coulomb interaction also includes an attractive exchange term, roughly proportional to Z, but this is usually quite negligible.

terms of this quantity must be ≥ 0. Since the expectation value calculated on an individual model represents an upper limit to the exact eigenvalue, the condition so obtained is in any case *necessary*. Now, all ordinary integrals (*11.11*–17) occurring in ${}^{\pi}\overline{V}^{\text{ord}}_{mt} + {}^{\alpha}\overline{V}^{\text{ord}}_{mt}$ (*11.11*–18) are positive, so that we finally get inequalities to be satisfied by the interaction parameters a. We take a nucleus consisting of equal numbers of neutrons and protons. If all spins are saturated, we get from formula (*11.11*–24), valid in such a case, the condition first enunciated by BREIT and FEENBERG [36*b*]

$$a_0 \geqslant 0. \tag{1}$$

But, as pointed out by KEMMER [37*b*], we must also prevent the configurations in which all spins are parallel from exhibiting non-saturation binding; from table *11.11*–1, considering that $\left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\right) = \left(\begin{smallmatrix}\uparrow\\\cdot\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\\cdot\end{smallmatrix}\right) = \left(\begin{smallmatrix}\uparrow\\+\end{smallmatrix}\middle|\begin{smallmatrix}\uparrow\\-\end{smallmatrix}\right)$ in our case, we read off the corresponding condition

$$a_0 + a_\sigma \geqslant 0. \tag{2}$$

We might further consider "nuclei" containing only neutrons. For such "neutron clusters", table *11.11*–1 yields the instability conditions:

(*a*) spins saturated (BREIT and FEENBERG [36*b*]):

$$a_0 + a_\tau \geqslant 0; \tag{3}$$

(*b*) spins parallel (FEENBERG [37*c*]):

$$a_0 + a_\tau + a_\sigma + a_{\sigma\tau} \geqslant 0. \tag{4}$$

But these conditions are weaker than (2). In the first place, from (2) and the first formula (*8.2*–12), we get the inequality

$$a_0 + a_\tau + a_\sigma + a_{\sigma\tau} \geqslant \tfrac{1}{3}, \tag{5}$$

which implies (4). The condition (5), first noticed by FEENBERG [37*c*], gets a simple interpretation when compared with the definition (*8.2*–9) of ${}^3p$: it means that the effective interaction in states of the type ${}^3P$ is necessarily a repulsion, amounting to at least $\frac{1}{3}$ of the absolute value of the ${}^3S$ attraction. As we have seen (*8.1, 8.33*), the proton-proton and proton-neutron scattering experiments are as yet inconclusive as regards Feenberg's prediction, although the latest results (*7.131*) would seem to fit it quite nicely. As to inequality (3), it follows from (2) and from the relation

$$a_\tau - a_\sigma = \frac{1-q}{4} > 0, \tag{6}$$

derived from (*8.2*–12) together with $q < 1$.

In this connexion, it may be observed that the instability of neutron clusters, even when the like-particle forces are assumed to be equal to the proton-neutron forces in all common configurations, is essentially a consequence of (*a*) the exclusion principle, (*b*) the fact that the attraction between two nucleons is smaller in the ${}^1S$ than in the ${}^3S$ configuration ($q < 1$) *, (*c*) the saturation character of the forces. Indeed, since the

* It is not necessary that (as is actually the case) there should be no bound state in the ${}^1S$ configuration: the "dineutron" might be stable, provided its binding energy be smaller than that of the deuteron.

exclusion principle prevents an accumulation of the neutrons in the lowest nucleon states to the same degree as in a "normal" nucleus, the lowest configuration of the neutron cluster (fig. *11.21*, (*a*)) has rather to be compared with a more loosely bound configuration, such as (*b*) in fig. *11.21*, of the normal nucleus. The comparison is simplest when the latter configuration is one in which all spins are parallel: for the only difference

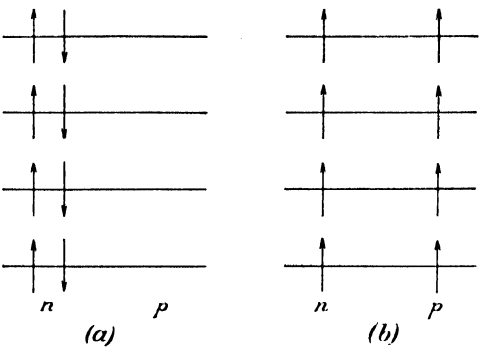

Fig. **11.21**. Comparison of neutron cluster (*a*) with normal nucleus (*b*). Symbolical representation as in fig. *10.14*.

in the interactions between pairs of nucleons is then that those of the singlet type (necessarily even) in (*a*) are replaced in (*b*) by interactions of the triplet type. But because of $q < 1$, this means that when going over from (*b*) to (*a*), a repulsion between neutrons with opposite spins is superposed on the forces present in configuration (*b*). This repulsion, then, being quadratic in the neutron number, becomes preponderant, and cannot be compensated by any attraction, since the latter would also be present in configuration (*b*) and imply a non-saturation binding in this configuration.

Another important consequence of Kemmer's condition (2) is to exclude any nuclear potential energy consisting solely of ordinary central forces, even with spin dependence: such a potential would correspond to (*8.2*–1,4,5)

$$a_\tau = a_{\sigma\tau} = 0, \quad \text{or} \quad M = H = 0,$$

so that (*8.2*–12) would give

$$a_0 + a_\sigma = -1,$$

in contradiction with (2). This means that forces yielding the type of saturation binding exactly similar to chemical binding are not sufficient, because they would not lead to an attraction in a triplet configuration: it is a well-known feature of the theory of chemical valency that the largest binding is obtained just between two particles with antiparallel spins. It is therefore necessary to introduce forces of the exchange type involving (at least implicitly) a dependence on the isotopic variables, as was done for the first time by Heisenberg and by Majorana. *In contrast with chemical bonds, brought about by spin exchange between the valency electrons, nuclear bonds also depend essentially on an exchange of charge between protons and neutrons.* From the point of view of a field theory of nuclear

interaction, it is interesting to note that this conclusion from a very general, qualitative argument entails the existence of charged nuclear fields, and rules out any theory involving *only* neutral fields (*8.31*).

As is well-known, — and can be verified by means of (*8.2*–12) together with (1), — a pure Heisenberg force ($a_0 = a_\tau$, $a_\sigma = a_{\sigma\tau} = 0$) leads to an attraction in the ^3S configuration and to a repulsion in the ^1S configuration: a situation ruled out by the fact that the complete saturation of the nuclear bonds is first attained in the α-particle, and not in the deuteron. It was just this argument that led Majorana to suggest his type of potential ($a_0 = a_\tau = a_\sigma = a_{\sigma\tau}$). But again, a pure Majorana force will not do, because it does not give any splitting of the ^3S and ^1S levels ($q = 1$).

It is not obvious that the necessary conditions (1), (2) should also be sufficient for the fulfilment of the first saturation requirement. Actually, it can be proved, under certain conditions, that this is the case when the additional inequality

$$a_{\sigma\tau} \geqslant 0 \tag{7}$$

is assumed. We shall now proceed to the proof of this statement. So far as the determination of the a's is concerned, however, we shall not need the sufficiency theorem: the second saturation requirements, combined with the necessary conditions (1) and (2), will lead to a complete determination of the a's.

11.22. *Sufficient saturation conditions. Conditions* (1), (2) *and* (7) *are, at least for an extensive class of short-range potential functions* $J(r)$, *sufficient for the fulfilment of the first saturation requirement.* We have thus to show that in virtue of these conditions the non-saturation part of the nuclear potential energy $\overline{V}$ will not be negative. The proof will be carried out in two steps: we shall first prove the inequality just mentioned on the long range fiction (*10.31*), i.e. treating J as a constant; since, however, the long range approximation of $\overline{V}$ is larger than its true value, this result must be completed by showing that an equivalent inequality still holds when the distance dependence of $J(r)$ is taken into account.

The first step is quite straightforward *. The long range approximation of $\overline{V}$ is given by (*10.33*–*28*) in terms of the multiplet quantum numbers. Let us put

$$s = S/A, \quad \tau = T/A, \quad p = P/A, \quad p' = \ldots; \tag{8}$$

we have thus to show that

$$\tfrac{1}{2}\, a_0 + 2 s^2 a_\sigma + 2\tau^2 a_\tau + 2\,(p^2 + p'^2 + p''^2 - s^2 - \tau^2)\, a_{\sigma\tau} \geqslant 0. \tag{9}$$

We begin by looking for the values of s, τ, p, ... which minimize the

* We follow here essentially the discussion given by BREIT and WIGNER [38*b*]; it is, however, much shortened due to the use of the parameters a instead of the W, B, H, M.

expression on the left: it will then suffice to prove the inequality for the minimum value so obtained. Because of condition (7) and of the inequality (6), we have only to distinguish between the following three possibilities:

(*a*) $a_\tau > 0$, $a_\sigma > 0$; (*b*) $a_\tau > 0$, $a_\sigma < 0$; (*c*) $a_\tau < 0$, $a_\sigma < 0$.

In case (*a*), the minimum is reached for *

$$s = \tau = p = p' = p'' = 0$$

and condition (1) is sufficient for the validity of (9). In case (*b*), the supermultiplet quantum numbers being first kept fixed, we have to make s as large as possible, i.e. $s = p$, and τ as small as possible, i.e. $\tau = 0$; then we must take for p its largest value, $p = \frac{1}{2}$ (*10.14*–9), and for the other quantum numbers the smallest value $p' = p'' = 0$. The left-hand side of (9) becomes $\frac{1}{2}(a_0 + a_\sigma)$, which is ≥ 0 by condition (2). It will be observed that the configurations of minimum potential energy considered in the two preceding cases are just those from which the inequalities (1) and (2), respectively, were derived as necessary saturation conditions.

Case (*c*) requires a little more attention: the supermultiplet being fixed, we must, since $|a_\sigma| > |a_\tau|$, above all choose s as large as possible: $s = p$; the largest value of τ compatible with this is then $\tau = p'$, so that the left-hand side of (9) reduces to

$$\tfrac{1}{2}a_0 + 2p^2 a_\sigma + 2p'^2 a_\tau + 2p''^2 a_{\sigma\tau}.$$

The choice $p = p' = \frac{1}{2}$, $p'' = 0$, which would correspond to independent variations of p, p', p'' is not acceptable, however, because it is not always compatible with the restriction (*10.14*–10) imposed on the parity of $P + P' + P''$. But, turning back to formulae (*10.14*–4, 8), we see that by decreasing Λ_4 and increasing Λ_1 by the same amount, we increase both P and P' without modifying either A or P''; we may therefore restrict ourselves to the cases for which $\Lambda_4 = 0$, i.e. $p'' = p + p' - \frac{1}{2}$. Taking then the largest value $p = \frac{1}{2}$, we have $p'' = p'$. For the latter, we must take the largest or smallest possible value according as $a_\tau + a_{\sigma\tau}$ is < 0 or > 0. Since condition (2), combined with the first formula (*8.2*–12), implies the last inequality, we have to choose $p' = p'' = 0$ and find that condition (2) is sufficient, as in case (*b*).

The result of the preceding discussion can be formulated by saying that, provided conditions (1), (2) and (7) are fulfilled, the mean long range potential energy in any multiplet state satisfies an inequality of the form

$$\text{av}\, \mathcal{V}_{\text{long range}} \geq -A\,\alpha J, \tag{10}$$

in which the numerical factor α is positive; according to (*10.33*–28), we may in fact take

$$\alpha = \begin{cases} \frac{1}{2}\,[a_0 + 3a_\sigma + 3\,(a_\tau + 3a_{\sigma\tau})] & \text{if } a_\sigma > 0 \\ \frac{1}{2}\,[a_0 + \ \ a_\sigma + \ \ a_\tau + \ \ a_{\sigma\tau}\,] & \text{if } a_\sigma < 0: \end{cases} \tag{11}$$

* We may, of course, assume for simplicity A to be even.

for it is clear that a can be computed, in each of the cases (a), (b), (c), for the special configuration which also minimizes the quadratic terms *. The inequality (10) means that the average interaction av $\mathcal{V}_{\text{long range}}$ does not yield a larger binding than the saturation effect represented by the right-hand side. The second step of the proof of the sufficiency theorem, indicated by WIGNER [36], will accordingly consist in showing that a similar inequality:

$$\overline{\mathcal{V}} \geqq - A\,a\,J^* \tag{12}$$

with a suitable constant J^*, still holds for the expectation value $\overline{\mathcal{V}}$ of the nuclear energy when the constant potential J is replaced by a function of distance $J(r)$ representing a short-range interaction. Actually, the theorem will be established for a potential of the general form

$$J(r) = \int_0^\infty g(\varrho)\, f(r/\varrho)\, d\varrho$$

$$\text{with } f(x) = \begin{cases} \frac{4\pi}{3}\left(1 - \frac{3}{4}x + \frac{1}{16}x^3\right) & \text{for } x \leqq 2 \\ 0 & \text{for } x \geqq 2. \end{cases} \tag{13}$$

In this formula $f(x)$ represents the volume of the space common to two spheres of unit radius, whose centres are at a distance x from each other, while $g(\varrho)$ is an arbitrary, nowhere negative, integrable function. It is clear that the function $f(r/\varrho)$, with fixed ϱ, describes a special kind of limited range interaction; a superposition such as (13) of such potentials of various ranges 2ϱ and strengths $g(\varrho)$ should at any rate represent an extensive class of short-range interactions.

Let us first consider an interaction of the special form $g(\varrho)\, f(r/\varrho)$ with a fixed value of ϱ. We have to compute

$$\overline{\mathcal{V}}_\varrho = \text{av}\ \tfrac{1}{2}\, g(\varrho) \sum_{i,k} f\left(\frac{r^{(ik)}}{\varrho}\right) \mathcal{O}^{(ik)}, \tag{14}$$

$\mathcal{O}^{(ik)}$ being the operator (*8.2–1*). Owing to the geometrical meaning of $f(x)$, we get, for any given configuration of the space coordinates $\vec{x}^{(i)}$ of the nucleons, exactly the operator occurring in (14) by assuming that all nucleons within any sphere of radius ϱ have the potential energy

$$\tfrac{1}{2}\, g(\varrho) \frac{1}{\varrho^3} \sum_{i,k}{}^{(\varrho)} \mathcal{O}^{(ik)} \tag{15}$$

(to be summed over all nucleons within the sphere), and integrating with respect to the coordinates of the centre of the sphere over all space: for the pair (i, k) will then contribute the interaction $g(\varrho)/\varrho^3 \cdot \mathcal{O}^{(ik)}$ when, and

---

* The value of a found for case (b) may be replaced by that corresponding to case (c); it is positive in virtue of (5), which is a consequence of (2).

only when, the centre of the sweeping sphere is situated in the space common to the spheres of radius ϱ and centres $\vec{x}^{(i)}$, $\vec{x}^{(k)}$, the volume of which space is just $\varrho^3 f(r^{(ik)}/\varrho)$. In order now to perform the integration with respect to the centre of the sweeping sphere, it is convenient to divide

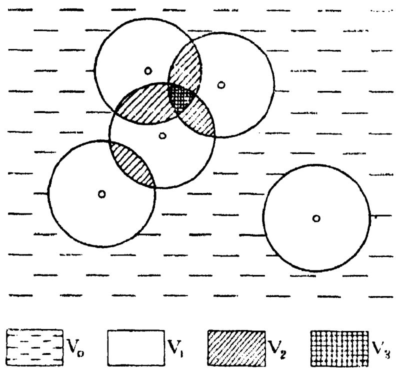

Fig. **11.22.** Division of space in regions V_s (see text).

all space into $A+1$ regions, called V_s $(s=0,1,2,\ldots,A)$, such that when the centre of the sweeping sphere is in V_s, this sphere contains s nucleons (see fig. *11.22*); obviously

$$\sum_{s=1}^{A} s\,V_s = A\cdot\frac{4\pi}{3}\varrho^3. \tag{16}$$

But for any position of the centre of the sphere within V_s, we are dealing with an interaction (15) of the "long range" type between the s nucleons, the expectation value of which, according to (10) *, is not smaller than

$$-s\,a\,\frac{g(\varrho)}{\varrho^3}.$$

The total interaction, averaged over spin and isotopic variables, is therefore $\geqslant -a\dfrac{g(\varrho)}{\varrho^3}\sum\limits_{s=1}^{A} s\,V_s$, i.e., by (16), $\geqslant -\frac{4\pi}{3}a g(\varrho)A$. Averaging over the space coordinates $\vec{x}^{(i)}$ does not alter this result, so that

$$\overline{V}_\varrho \geqslant -\frac{4\pi}{3}\,a\,g(\varrho)\,A. \tag{17}$$

Integrating (17) over ϱ, we indeed get (12), with

$$J^* = \frac{4\pi}{3}\int_0^\infty g(\varrho)\,d\varrho = J(0), \tag{18}$$

according to (13).

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* At this stage of the argument, we are dealing with the expectation value in a state corresponding to fixed values of the $\vec{x}^{(i)}$; but inequality (10) also holds for such a state, since it can be represented by a superposition of multiplet states.

11.3. The second saturation requirement

11.30. The second saturation requirement will be more difficult to formulate by our approximate methods of calculation of the energy of nuclear states, since its fulfilment depends on a balance between the various contributions to this energy, which will partly depend on the choice of the nuclear potential, but partly also on the accuracy of the method. We must therefore not expect as clear-cut conditions as those expressing the first requirement. Nevertheless, we owe to VOLZ [37] a formulation of the second requirement which permits a rather accurate determination of the interaction parameters a.

11.31. *Light α-nuclei.* Let us first consider some light "α-nucleus" (*9.22*), such as $^{16}_{8}O$. If we calculate its binding energy by the quasi-atomic method (*9.21*), the average nuclear energy will be of the form (*11.11*–24), so that we can expect from the formulation of the second saturation requirement in this case further information on the parameter a_0. If one chooses a nuclear potential of the Gauss type and takes as trial wave-functions of the individual nucleon states the eigenfunctions of the spatial harmonic oscillator, the interaction integrals I and X, as well as the mean kinetic and Coulomb energies, can be calculated explicitly. Adopting definite numerical values for the range of force, the potential strength and the ratio q, the variational procedure can be carried out for different values of a_0. Since $I > X$, the calculated binding energy will decrease in absolute value as a_0 increases; by requiring that the quasi-atomic approximation should at any rate be sufficiently good, for this nucleus, to give the right sign of the binding energy, one therefore obtains an *upper limit* α_0 for a_0.

Volz took $\varkappa^{-1} = 2{,}17 \cdot 10^{-13}$ cm, $J = 37{,}9$ MeV, $q = 0{,}5$; but the use of better values (*6.432*) should not alter the order of magnitude of his result. He finds $\alpha_0 = 0{,}15$ and states that even a slight sharpening of the condition imposed on the calculated binding energy leads to a considerable smaller value of α_0. We thus get a condition of the form

$$a_0 \leqslant \alpha_0, \qquad \alpha_0 \ll 1. \tag{1}$$

The comparison of the limitations (*11.21*–1) and (1) undoubtedly points to the determination

$$a_0 = 0 \tag{2}$$

as the most likely one; a small positive value of a_0 can, however, not be definitely excluded.

11.32. *Heavy nuclei.* We next turn to a study of heavy isobaric nuclei (of mass number 200, say) with varying neutron excess; more specifically, we consider even nuclei in states with saturated nucleon spins. From table *11.11*–2, applicable to this case, we see that we have to do

with the combinations of parameters $a_0 + a_\tau$ and $a_0 - a_\tau$, or more simply, if we adopt (2), with the parameter a_τ only. It is indicated here to use the Fermi gas model, the nuclear radius being regarded as a variational parameter; all integrals occurring in the table *11.11*–2 can again be calculated explicitly if the potential is of the Gauss type. But we cannot be confident that the calculated binding energy $\mathcal{E}$ will give a sufficiently exact idea of the value $\mathcal{E}'$ which should actually correspond to the nuclear potential adopted. To remedy this situation, Volz introduces the ratio

$$\varepsilon = \frac{|\mathcal{E}'| + K + G}{|\mathcal{E}| + K + G} \tag{3}$$

of actual to calculated nuclear potential energy (*11.13*–34) and tries in each case so to adjust the variational parameter R as to get the minimum value ε_0 of this ratio, which he calls the "error coefficient". The solution so obtained is the best one of the type considered which yields the value $|\mathcal{E}'|$ for the binding energy when the nuclear potential is ε_0 times the assumed value. Since we always have $|\mathcal{E}| < |\mathcal{E}'|$, the error coefficient ε_0 will be ≥ 1. But if we now impose the value $|\mathcal{E}'|$ which we wish to obtain, the ε_0 resulting from a certain choice of potential could become < 1: this would mean, then, that this choice is inadequate. So ε_0 furnishes some measure, not only of the accuracy of the method of approximation, but also of the adequacy of the nuclear potential adopted *.

Coming back to the even isobars of mass number 200, the error coefficient can be calculated for different values of the neutron excess and various choices of the parameter a_τ; for $|\mathcal{E}'|$, one may take throughout the value 1,7 MU, corresponding to the (probable) mass-defect of the stable nucleus $^{200}_{80}Hg$. For each value of a_τ, the stablest nucleus is that with the smallest error coefficient, so that we get an idea of the variation with a_τ of the neutron excess of the stablest nucleus of mass number 200. Volz' result is represented by fig. *11.32*, the main features of which should

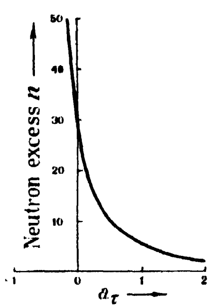

Fig. 11.32. Neutron excess n of stablest nucleus of mass number 200 in terms of interaction parameter a_τ.

remain valid if better potential constants were used. As expected, negative a_τ's favour large neutron excesses, due to predominance of non-saturation

* The calculation of the error coefficient in suitably chosen cases may therefore also serve to derive limitations of the a's imposed by the *first* saturation requirement. Thus Volz deduced in his paper conditions practically equivalent to (*11.21*–1,3).

attraction: but with increasing a_τ, the neutron excess diminishes rapidly. The empirical value ($n \approx 40$) would correspond, according to fig. *11.32*, to a negative a_τ of very small absolute value, but we cannot expect the method to yield such a precise determination of a_τ. We can at most exclude such a_τ's as would lead to neutron excesses smaller than, say, half the empirical value. This gives an upper limitation of a_τ:

$$a_\tau \lessapprox \alpha_\tau, \qquad \alpha_\tau \approx 0{,}1 \tag{4}$$

(Volz gets $\alpha_\tau = 0{,}12$). According to (*11.21*–6), this condition implies an analogous limitation for a_σ:

$$a_\sigma \lessapprox \alpha_\sigma, \qquad \alpha_\sigma \approx 0 \tag{5}$$

(in fact, with $q \approx 0{,}6$, the upper limit $\alpha_\sigma = \alpha_\tau - \frac{1}{4}(1-q)$ becomes nearly 0). It would seem that a natural choice of the parameter a_σ, just compatible with (5) and with Kemmer's condition (*11.21*–2), combined with (2), would be, as proposed by KEMMER [37*b*],

$$a_\sigma = 0. \tag{6}$$

On the contrary, the determination $a_\tau = 0$, originally proposed by Volz, is in contradiction with Kemmer's condition, on account of (2) and (*11.21*–6). If, however, we combine Kemmer's condition with the inequality (1) rather than the equality (2), we cannot exclude the possibility of a small positive value for a_σ.

11.33. *Determination of interaction parameters.* Once we adopt, in accordance with (2) and (6),

$$a_0 = a_\sigma = 0, \tag{7}$$

the values of a_τ and $a_{\sigma\tau}$ follow from (*8.2*–12):

$$\begin{aligned} a_\tau &= \tfrac{1}{4}(1-q) \approx 0{,}1 \\ a_{\sigma\tau} &= \tfrac{1}{12}(1+3q) \approx 0{,}23. \end{aligned} \tag{8}$$

The final form of the nuclear potential operator is thus

$$V_{\text{nucl}} = \boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\,[a_\tau + a_{\sigma\tau}\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}]\,J(r^{(12)}), \tag{9}$$

the constants a_τ, $a_{\sigma\tau}$ being given by (8), whereas the analytical form of $J(r)$ remains unsettled; it must be remembered that a small additional potential of the form

$$V'_{\text{nucl}} = [a_0 + a_\sigma\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}]\,J(r^{(12)}), \tag{10}$$

likewise with positive a_0, a_σ, would still be admissible. It will be recalled that the form (9) is just that corresponding to a symmetrical meson theory (*8.32*–17), whereas (10) corresponds to a neutral theory (*8.32*–16).

It is certainly remarkable that in spite of the inaccuracies affecting the preceding discussion, its outcome should receive such a simple interpretation on meson field theory.

According to (*8.2*–5), the potential (9) actually corresponds to a mixture of the four types of exchange potentials (*4.341*–34) in proportions given by

$$\begin{aligned} W &= \tfrac{1}{6}(3q-1) \approx 0{,}13 \qquad & M &= -\tfrac{1}{3}(3q+1) \approx -0{,}93 \\ B &= \tfrac{1}{2}M \approx -0{,}46 \qquad & H &= 2W \approx 0{,}26: \end{aligned} \tag{11}$$

the most important term is thus a Majorana potential, but the Bartlett term is also of considerable weight. The effective potentials for odd states, derived from the operator (9), have already been discussed in *8.32*; they are repulsive both for singlet and for triplet configurations. In fact, according to (*8.2*–9), we find

$$-{}^1p = 3q \approx 1{,}8, \tag{12}$$

while $-{}^3p$ takes its lowest possible value (*11.21*)

$$-{}^3p = \tfrac{1}{3}. \tag{13}$$

11.4. Stability of isobars

Quite independently of the saturation properties of the nuclear bindings, another set of conditions limiting the permissible values of the interaction parameters a can be derived from the consideration of isobaric nuclei. However, these conditions are obtained only in the very rough approximation corresponding to the long range fiction (*10.31*) and are consequently not as reliable as the saturation conditions just discussed. Still, it will be worth while to examine them too in order to verify their compatibility with the results arrived at in the preceding section.

We have seen (*3.21*) that odd nuclei heavier than ^{14}N are unstable, whereas the neighbouring even isobars can both be stable. According to the stability condition *1.131*, this situation is expressed by the inequalities

$$W[N,Z] < W[N+1,Z-1], \quad W[N+1,Z-1] > W[N+2,Z-2], \tag{1}$$

if $[N, Z]$ represents the composition of one of the even isobars. The inequalities (1) imply the further one

$$W[N,Z] - W[N+1,Z-1] < \tfrac{1}{2}\{W[N,Z] - W[N+2,Z-2]\}, \tag{2}$$

which, if we neglect unimportant terms arising from the variations of kinetic and Coulomb energies, reduces to an inequality involving only nuclear potential energies:

$$\overline{\mathcal{V}}[N,Z] - \overline{\mathcal{V}}[N+1,Z-1] < \tfrac{1}{2}\{\overline{\mathcal{V}}[N,Z] - \overline{\mathcal{V}}[N+2,Z-2]\}. \tag{3}$$

The average nuclear energies of the ground states of the even isobars are immediately given, on the individual model approximation, by the formulae (*11.11*–7, 21) with the integrals and coefficients given by table *11.11*–2. Introducing the long range approximation in the definition

(*11.11*–20) of the integrals, and taking account of the orthogonality of the spatial wave-functions φ_n, one finds that these integrals reduce to integral multiples of the constant potential J, specified as follows:

m_t	$\bar{V}^{\text{ord}}_{m_t}$	$\bar{V}^{\text{exch}}_{m_t}$
$+1$	number of neutron pairs	number of neutron bonds *
-1	number of proton pairs	number of proton bonds
0	number of neutron-proton pairs	number of neutron-proton bonds

* The concept of *bond* is defined in *10.31*. (4)

The calculation of the different terms of $\bar{V}$ for a given configuration thus becomes a simple matter of enumeration (INGLIS and YOUNG [37]) *.

The treatment of the odd isobar requires some attention, because the supermultiplet of the ground state (*10.331*) splits under the influence of the spin dependent forces into a singlet and a (degenerate) triplet term, and we have to calculate the energies of these two terms. For this purpose, we consider the two distinct configurations ((*a*), (*b*) in fig. *11.4*) for

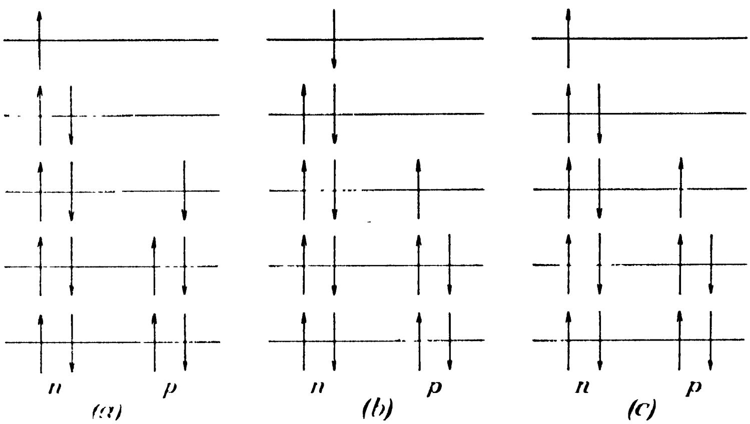

Fig. 11.4. Lowest configurations of odd isobars:
(*a*), (*b*) resonating configurations with $S_z = 0$;
(*c*) configuration $S_z = 1$.
Symbolic representation explained in fig. *10.14*.

which the z-component of the spin $S_z = 0$: between these two configurations, which differ by the orientations of the spins of the odd proton and neutron, there will be a "resonance" corresponding to the separation of singlet and triplet stationary states. In other words, the eigenfunctions of these two states will be linear combinations of the Slater-determinants describing the resonating configurations, and we have to solve a secular

* The integrals of table *11.11*–1 can be expressed in the same way; one has simply to add to the numbers occurring in (4) the qualification "with parallel spins" or "with antiparallel spins", according as one has to do with a ${}^{\pi}\bar{V}$ or a ${}^{\alpha}\bar{V}$.

equation of the second order to find the eigenvalues of $\mathcal{V}$. The diagonal elements are immediately given by the same argument as above, and have obviously the same value $\overline{\mathcal{V}}_{\text{diag}}$ in both configurations; as to the non-diagonal elements, an easy calculation yields, on the long range approximation, the common value $2\mathcal{J}(a_\sigma - a_{\sigma\tau})$. Therefore

$$\overline{\mathcal{V}}[N+1, Z-1] = \overline{\mathcal{V}}_{\text{diag}}[N+1, Z-1] \pm 2\mathcal{J}(a_\sigma - a_{\sigma\tau}). \tag{5}$$

The corresponding eigenfunctions are proportional (as expected) to the sum and difference, respectively, of the Slater-determinants; the difference pertains to the singlet, the sum to the triplet state. It is readily verified that the direct calculation of the mean nuclear energy in the other triplet sub-states, corresponding to $S_z = \pm 1$, yields the same result as for the triplet sub-state with $S_z = 0$: in fact, the state with $S_z = 1$, e.g., is represented by a single Slater-determinant (configuration (c) in fig. *11.4*), and the comparison of configurations (*a*) and (*c*), keeping in mind the rules contained in (4) and the accompanying footnote, immediately shows that the difference between $\overline{\mathcal{V}}(\text{config. (c)})$ and $\overline{\mathcal{V}}_{\text{diag}}(\text{config. (a)})$ is just $2\mathcal{J}(a_\sigma - a_{\sigma\tau})$.

For the differences occurring on both sides of inequality (3), one thus finally gets, if one further makes the natural assumption that the neutron excess n of the isobar $[N, Z]$ is positive,

$$\tfrac{1}{2}\{\overline{\mathcal{V}}[N,Z] - \overline{\mathcal{V}}[N+2, Z-2]\} = -2\mathcal{J}\{a_\tau(n+3) + 3a_{\sigma\tau}\} \tag{6}$$

$$\overline{\mathcal{V}}[N,Z] - \overline{\mathcal{V}}[N+1, Z-1] = -2\mathcal{J}\{[a_\tau(n+2) + a_\sigma + 5a_{\sigma\tau}] \pm (a_\sigma - a_{\sigma\tau})\},$$

the $\pm$-sign referring to the triplet and singlet state of the odd isobar, respectively. Expressing that condition (3) must hold as well for the singlet as for the triplet state of the odd isobar, we thus get the inequalities (BREIT and WIGNER [38*b*]).

$$3a_{\sigma\tau} - a_\tau > 0, \tag{7}$$

$$2a_\sigma - a_\tau + a_{\sigma\tau} > 0. \tag{8}$$

Since, by (*8.2*–12),

$$2a_\sigma - a_\tau + a_{\sigma\tau} = \tfrac{1}{3}(a_0 + a_\sigma) + \tfrac{1}{6}(3q - 1), \tag{9}$$

condition (8) can be deduced from Kemmer's condition (*11.21*–2), provided only $q < \frac{1}{3}$.

Inequality (7), on the contrary, is not reducible to our saturation conditions (*11.21*–1, 2). With the help of (*8.2*–12) (by forming the expression $3 + q$), it may be seen that, together with (*11.21*–1), it implies (*11.21*–7), so that (*11.21*–1, 2) and (7) together can be considered (*11.22*) as a set of necessary and sufficient conditions for the fulfilment of both the first saturation and odd isobar instability requirements. Condition (7) is, of course, fulfilled by our final choice (*11.33*–8) of the parameters a_τ, $a_{\sigma\tau}$; in terms of the exchange potential parameters (*8.2*–5), it takes the simple form

$$H > M. \tag{10}$$

Combining (7) with the second formula (*8.2*–12), one derives the further inequality

$$a_0 - 3a_\sigma + q > 0; \tag{11}$$

inserting (7) and (11) into the expression (*8.2*–9) of ${}^1p$, one finds

$${}^1p < q: \tag{12}$$

the ${}^1P$ effective potential is either a repulsion or an attraction smaller in absolute value than that in ${}^1S$ states. If one uses the equality (*11.31*–2) instead of the inequality (*11.21*–1), one gets further limitations on the p's: from (*8.2*–9, 12) and (*11.21*–2) one then derives that (FEENBERG [37*c*])

$${}^1p \geqslant -3q: \tag{13}$$

from (*8.2*–12) and (7), that (BREIT and WIGNER [38*b*])

$$-{}^3p < \tfrac{1}{9}(4q+3). \tag{14}$$

Grouping together (12) and (13), (14) and (*11.21*–5), we have then, if $a_0 = 0$,

$$\begin{aligned} \tfrac{1}{3} \leqslant -{}^3p &< \tfrac{1}{9}(4q+3) \\ -3q \leqslant {}^1p &< q. \end{aligned} \tag{15}$$

If further (*11.32*–6) $a_\sigma = 0$, $-{}^3p$ and ${}^1p$ reach, as we have seen (*11.33*–12, 13), their lower limits.

11.41. *Ground state of odd isobar.* With our choice of interaction parameters, $a_\sigma - a_{\sigma\tau} < 0$, so that, according to the estimate (5), the triplet state of the isobar lies lower than the singlet state. For the lightest odd nuclei (the only stable ones), calculations based on the quasi-atomic model (HUND [37]) confirm this conclusion, actually indicating the ground state to be a ${}^{13}S$ one. In the case of ${}^2H$, ${}^6Li$ and ${}^{14}N$, it has been ascertained that, in agreement with this result, the total angular momentum has the quantum number $J = 1$.

CHAPTER XII

PROPERTIES OF HEAVY NUCLEI

12.0. This Chapter contains a somewhat closer discussion of the Fermi gas model for heavy nuclei. It will be seen that in spite of its inadequacy with respect to quantitative determinations, this model yields a fairly comprehensive picture of the ground states of such nuclei. In the first approximation, various effects can be treated with a view to deriving a general expression for the binding energy, including the correction for surface tension and Coulomb interaction. The next approximation will be carried out only neglecting the latter effects, i.e. for the "standard heavy nucleus" (*9.41*), and besides furnishing a more accurate expression for the volume energy, will disclose in an interesting way the tendency to "α-clustering" already referred to above (*9.12*). Finally, the penetration of fast nucleons into a heavy nucleus will be investigated; in particular, a derivation will be given of the formula used in *7.14*.

12.1. Mass-defects and nuclear radii

12.11. *The volume energy.* In first approximation, we get the total volume energy of the nucleus $[N, Z]$ simply by adding to the kinetic energy given by (*9.41*–7) (with $g_s = 2$, corresponding to complete spin saturation) the average value of the nuclear energy, which can be calculated with the help of formulae (*11.11*–20, 21) and the accompanying table *11.11*–2. The individual wave-functions $\varphi_n(\vec{x})$ being plane waves:

$$\varphi_n(\vec{x}) = \frac{1}{\sqrt{V}}\, e^{\frac{i}{\hbar}\vec{p}_n\vec{x}}, \tag{1}$$

the integrals (*11.11*–20) occurring in the nuclear energy can ultimately be expressed in terms of certain Fourier coefficients of the nuclear potential $Jw(\varkappa r)$. For this purpose, let us first introduce the Fourier expansion of the law of force

$$w(\varkappa r) = \int w_p(p)\, e^{-\frac{i}{\hbar}\vec{p}\vec{x}}\, \frac{V}{(2\pi\hbar)^3}\, dv_p\,; \tag{2}$$

the Fourier coefficients are of the form

$$w_p(p) = \frac{1}{V}\int w(\varkappa r)\, e^{\frac{i}{\hbar}\vec{p}\vec{x}}\, dv = \frac{2\pi}{V\varkappa^3}\, F\left(\frac{p}{\hbar\varkappa}\right) \tag{3}$$

with

$$F(u) = \frac{2}{u}\int_0^\infty w(\xi)\,\xi\,\sin u\xi\, d\xi.$$

Next, we calculate the matrix-elements of the nuclear potential:

$$(\vec{p}_i^{(1)}\vec{p}_i^{(2)}\,|J(r)|\,\vec{p}_f^{(1)}\vec{p}_f^{(2)}) = \int \varphi^*_{\vec{p}_i^{(1)}}(\vec{x}^{(1)})\,\varphi^*_{\vec{p}_i^{(2)}}(\vec{x}^{(2)})\,J(r^{(12)})\,\varphi_{\vec{p}_f^{(1)}}(\vec{x}^{(1)})\,\varphi_{\vec{p}_f^{(2)}}(\vec{x}^{(2)})\,dv^{(1)}dv^{(2)}$$

$$= \delta(\vec{p}_i^{(1)}+\vec{p}_i^{(2)}-\vec{p}_f^{(1)}-\vec{p}_f^{(2)})\,J\,w_p(p_0), \tag{4}$$

with

$$\vec{p}_0 = \vec{p}_i^{(1)} - \vec{p}_f^{(1)}\ (= \vec{p}_f^{(2)} - \vec{p}_i^{(2)}).$$

With this notation *, the formulae (*11.11*–20) take the form

$$(+\,|\,-) = \begin{cases} \sum\limits^{(+)}_{\vec{p}^{(1)}} \sum\limits^{(-)}_{\vec{p}^{(2)}} (\vec{p}^{(1)}\vec{p}^{(2)}\,|\,J\,|\,\vec{p}^{(1)}\vec{p}^{(2)}) & \text{(ordinary)} \\ \sum\limits^{(+)}_{\vec{p}^{(1)}} \sum\limits^{(-)}_{\vec{p}^{(2)}} (\vec{p}^{(1)}\vec{p}^{(2)}\,|\,J\,|\,\vec{p}^{(2)}\vec{p}^{(1)}) & \text{(exchange).} \end{cases} \tag{5}$$

Accordingly, the ordinary integrals reduce to $Jw_p(0)$ times N^2, Z^2 or NZ, as the case may be, so that their total contribution to the energy, according to table *11.11*–2 in which (*11.31*–2) $a_0 = 0$. is proportional to the square of the neutron excess $(N-Z)^2$; and since $w_p(0)$ is inversely proportional to A, this contribution is, more precisely, proportional to $A\cdot(n/A)^2$, just like the analogous term in the kinetic energy. As regards the exchange integrals, which involve summations (or integrations) over momenta up to the maximum values $p_m^{(N)}$, $p_m^{(Z)}$ for neutron or proton states (*9.41*–1), we arrive at the same conclusion: for these integrals will ultimately be functions of the $p_m^{(N)}$, $p_m^{(Z)}$, which only depend on the ratios N/A, Z/A, i.e. on n/A. If, therefore, assuming that we have to do with an even nucleus of relatively small neutron excess, we expand such expressions in powers of n/A, the lowest power occurring (besides a term independent of n) will be the second, because the volume energy, as long as the Coulomb interaction is disregarded, is symmetrical with respect to its dependence on N and Z. We thus arrive, for the volume energy, at an expression of the form suggested by the empirical distribution of mass-defects, i.e. a mean energy per nucleon $-\varepsilon_1[1-\gamma(n/A)^2]$ (*2.21*–3).

12.12. *The surface energy.* The model hitherto considered does not give rise to any surface effects, since it corresponds to considering the nucleon assembly under the influence of a fictitious constant (and negative) potential abruptly rising to zero at the boundary of the domain occupied by the nucleons. In order to account for the surface energy, one may (B & B, § 29) replace the sharp rise of the potential by a gradual one. More definitely, let us call x a length measured along the normal to the

* The indices i, f refer to the "initial" and "final" state, respectively, of the transition to which the matrix-element is related.

boundary, and varying from a value 0 outside (i.e. where the potential takes the value 0) to a value l, at which the potential assumes its constant negative value; let us assume the potential to vary linearly with x throughout the surface layer. If the thickness l is large in comparison with the wave-lengths of the nucleons, i.e. $l \gtrsim d$, we can apply to any volume element of the layer the statistical arguments leading to a relation between the maximum value of the momentum at that point (depending on the fictitious potential) and the particle density: the latter quantity being proportional to the cube of the maximum momentum will therefore vary as $(x/l)^{3/2}$; calling ϱ_0 the constant value of the density inside the nucleus, we have thus, within the layer,

$$\varrho = \varrho_0 (x/l)^{3/2}. \tag{6}$$

The mean kinetic energy at x will be proportional to $\varrho^{2/3}$, so that the mean kinetic energy per unit volume will be proportional to $\varrho^{5/3}$. If the density were ϱ_0 throughout, without boundary layer, the volume of the nucleus would have a value V such that $\varrho_0 V$ is equal to the number of nucleons, and the kinetic energy would have a corresponding value K. From the density law (6), we easily derive the change of kinetic energy

$$\delta K = -\frac{4}{35} K \cdot \frac{lS}{V}, \tag{7}$$

S denoting the surface of the nucleus. We have just seen how the mean nuclear energy can be expressed in terms of the maximum momenta; we therefore know its expression as a function of x within the layer and are thus able to calculate the variation due to this layer. In this way, the total contribution from surface effects to the average energy can in principle be estimated. It would be pointless, however, to go into further details of such calculations, since we cannot expect them to yield reliable quantitative results.

It must still be observed that there is, besides the term (7), another surface contribution to the kinetic energy, which has nothing to do with the variation of the potential energy at the boundary: it simply arises from the discreteness of the eigenvalues of the momenta of the individual nucleons and can be estimated by performing the summations over the momenta to a greater accuracy than that obtained by replacing them simply by integrations. Such estimates, performed by FEENBERG [41c], yield, besides a main term of the form (9.41–7), $K^{(0)}A\left[1+\frac{5}{9}\left(\frac{n}{A}\right)^2\right]$, with $K^{(0)} = K(\frac{1}{2}d/r_0)^2$, a surface term $K^{(0)}A^{2/3}\cdot 1{,}92\left[1+\frac{2}{9}\left(\frac{n}{A}\right)^2\right]$ and an additional term $c_0 K^{(0)} A^{1/3}$ (with a numerical constant $c_0 \approx 1$), which apparently cannot admit of any simple geometrical interpretation *.

* See further a paper by VOLKOFF [42a], in which the influence of the shape of the nucleus on the kinetic energy is also discussed.

12.13. *Nuclear radii.* The expression finally obtained by adding the volume and surface contributions from kinetic and nuclear energy, as well as the Coulomb interaction between the protons, shows the expected dependence on the proton and neutron numbers. It remains to be seen whether at least the order of magnitude of the different terms comes out about right. If one assumes a definite distance dependence for the nuclear potential, with given numerical values of the strengths and range, the only remaining parameter is the nucleon radius r_0 (it also occurs in the potential energy terms through the maximum momenta). We may then look for the "best" value of r_0, i.e. that which minimizes the total energy.

It proves convenient to take as variational parameter the dimensionless quantity

$$\mathrm{r} = \tfrac{1}{2}\,(9\pi)^{1/3}\,(\varkappa r_0)^{-1} = \frac{1.523}{\varkappa r_0}; \tag{8}$$

the maximum momenta can then simply be expressed as

$$p_m^{(N)\text{ or }(Z)} = p_m^{(0)}\left(1 \pm \frac{n}{A}\right)^{1/3}, \tag{9}$$

with

$$p_m^{(0)} = \hbar\varkappa\,\mathrm{r}, \tag{10}$$

so that all terms arising from the nuclear interaction immediately appear as functions of r. The mean total energy per nucleon has therefore the general form

$$\varepsilon(\mathrm{r}) = -\,\varepsilon_1(\mathrm{r})\,[1-\gamma(\mathrm{r})\,(n/A)^2] + \sigma(\mathrm{r})\,A^{-1/3} + \varepsilon_0\,\mathrm{r}\,A^{-4/3}\,Z^2, \tag{11}$$

ε_0 being a constant. Strictly speaking, the best value r should be found by solving the equation $\varepsilon'(\mathrm{r}) = 0$, i.e.

$$\varepsilon_1'(\mathrm{r}) = [\varepsilon_1\gamma]'\,(n/A)^2 + \sigma'\,A^{-1/3} + \varepsilon_0 A^{-4/3}\,Z^2. \tag{12}$$

As a first approximation, we may take the value r_0 for which

$$\varepsilon_1'(\mathrm{r}_0) = 0: \tag{13}$$

this corresponds to a constant density of nuclear matter, irrespective of the mass number. Since, on account of (13),

$$\varepsilon_1'(\mathrm{r}) = (\mathrm{r}-\mathrm{r}_0)\,\varepsilon_1''(\mathrm{r}_0), \tag{14}$$

we get the next approximation by inserting in (14) the value (12) of $\varepsilon_1'(\mathrm{r})$, in which we may replace r by r_0. This gives

$$\mathrm{r}_1 = \mathrm{r}_0\left\{1 - \frac{1}{9MC^2}\left[(\varepsilon_1\gamma)'_{\mathrm{r}_0}\left(\frac{n}{A}\right)^2 + \sigma'(\mathrm{r}_0)\,\mathrm{r}_0 A^{-1/3} + \mathcal{G}(\mathrm{r}_0)\right]\right\}; \tag{15}$$

in this formula, $\mathcal{G}(\mathrm{r}_0)$ denotes the value of the Coulomb energy for $\mathrm{r} = \mathrm{r}_0$ while C represents the velocity of sound in nuclear matter, referred to the

velocity of light as unit. This quantity is indeed defined, ϱ being the density, by

$$C^2 = \frac{1}{M} \varrho^2 \frac{d^2(-\varepsilon_1)}{d\varrho^2} = \frac{1}{9M} r_0^2 \frac{d^2(-\varepsilon_1)}{dr_0^2}; \tag{16}$$

the last equality follows from the preceding one because of $\varrho \sim r_0^3$ and (13). Relation (15), which has been derived in a less complete form by FEENBERG [41*a*] and as above by PRESENT [41], shows that one would expect a slight increase of the nucleon radius r_0 with increasing mass number. In fact, the three correction terms in (15) all work in the same sense: the increase of the relative neutron excess with increasing A means looser binding and smaller density; the contraction of the nucleus due to surface forces becomes smaller in heavier nuclei, again resulting in a decrease of the density; and the Coulomb repulsion of the protons clearly produces the same effect.

Explicit calculations according to the above scheme have been carried out most extensively with the Gauss potential. On the determination of r_0 from (13) and of $\varepsilon_1(r_0)$ we shall say more in the next section, after we have calculated the next approximation to ε_1: for the first approximation turns out to be completely inadequate. Even then, it will appear that a set of values of J and $\varkappa$ consistent with the empirical values of r_0 and ε_1 would correspond to a much tighter binding than indicated by the analysis of the two-nucleon system. Adopting such values for J and $\varkappa$, we find that to account for the empirical value of the surface tension (*2.21*–5), we should have to take $l \approx 4{,}8\, r_0$, an improbably large value. Finally, Present's estimate * of the correction terms in formula (15) leads to the expression for the radius

$$r_0' = r_0 \left[1 + 0{,}8 \left(\frac{n}{A} \right)^2 - \frac{0{,}3}{A^{1/3}} + 0{,}010 \frac{Z^2}{A^{4/3}} \right]; \tag{17}$$

the factor between brackets increases by 9 % from ^{56}Fe to ^{207}Pb.

12.14. *Variation of density within a nucleus.* The Coulomb repulsion between protons must obviously give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the centre to a maximum near the boundary. A non-uniform proton distribution may be expected to create forces which distort the neutron distribution and tend to make the two particle densities vary in the same manner. This density variation has been analysed by FEENBERG [41*b*] neglecting surface effects. It is then permissible to assume that the deviations from uniformity of the proton and neutron distributions are small. The only further assumption that must be made is that the total volume energy may be written as an integral $\int \mathcal{V}(\varrho^{(n)}, \varrho^{(p)})dv$, the integrand being a function of

* Present's treatment differs slightly from the above in some particulars; but it is not worth while working out the resulting modifications in formula (17).

the neutron and proton densities $\varrho^{(n)}$, $\varrho^{(p)}$ only. Besides this volume energy, there occurs a correction term of kinetic energy, which is easily found to be

$$\frac{\hbar^2}{2M}\int\left[\frac{(\text{grad}\,\varrho^{(n)})^2}{\varrho^{(n)}}+\frac{(\text{grad}\,\varrho^{(p)})^2}{\varrho^{(p)}}\right]dv\,, \tag{18}$$

and the Coulomb energy, which, including the exchange term calculated by the statistical method, becomes

$$G=\tfrac{1}{2}\cdot\frac{Z-1}{Z}e^2\int\frac{\varrho^{(p)}(\vec{x})\,\varrho^{(p)}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,dv\,dv'-\left(\frac{81}{64\pi}\right)^{1/3}e^2\int(\varrho^{(p)})^{4/3}\,dv\,. \tag{19}$$

The variational method may now be applied, using for $\varrho^{(n)}$, $\varrho^{(p)}$ trial functions which are chosen in the form of simple polynomials in the radius vector r, satisfying the boundary conditions

$$\frac{d\varrho^{(n)\text{ and }(p)}}{dr}=0 \quad \text{for} \quad r=R \tag{20}$$

as well as the normalization conditions

$$\int_{(\text{nucl})}\varrho^{(n)}\,dv=N \quad , \quad \int_{(\text{nucl})}\varrho^{(p)}\,dv=Z\,. \tag{21}$$

The final result is that both densities are found to vary in a parallel manner, tending toward the formation of a hollow centre within the nucleus. For the nucleus $^{200}_{80}\text{Hg}$, e.g., one finds

$$\frac{\varrho^{(n)}(R)}{\varrho^{(n)}(0)}=1{,}20 \quad , \quad \frac{\varrho^{(p)}(R)}{\varrho^{(p)}(0)}=1{,}43\,. \tag{22}$$

The resulting changes in energy and nuclear radius are quite insignificant: the change in total energy of $^{200}_{80}\text{Hg}$ is only about —7 MeV.

12.2. Standard heavy nucleus to second approximation

12.20. In carrying the approximation now a step further, we shall disregard all surface effects as well as Coulomb energy, which means that we consider the ground state of the "standard heavy nucleus", as defined in *9.41* (except for the fact that we provisionally leave the value of the nucleon radius r_0 undetermined). We shall begin by deriving a more accurate value of the mean volume energy per nucleon ε_1; we shall then discuss more closely the distribution of the different nucleons in this state, first as regards their momenta, then as regards their positions and spins. Especially in the latter respect interesting results are obtained.

12.21. *The volume energy.* In the calculation of the energy, we have been applying the usual perturbation method, using as a perturbation parameter the strength of the nuclear potential. In the expansion

$$\mathcal{E}=\mathcal{E}^{(0)}+\mathcal{E}^{(1)}+\mathcal{E}^{(2)}+\ldots \tag{1}$$

in powers of this parameter, the initial approximation $\mathcal{E}^{(0)}$ is simply the unperturbed kinetic energy; the next one, $\mathcal{E}^{(1)}$, is the first approximation (the only one hitherto considered) to the average value $\overline{\mathcal{V}}^{(1)}$ of the potential energy, while the second, $\mathcal{E}^{(2)}$, consists of contributions to kinetic as well as to potential energy, $\mathcal{K}^{(2)}$ and $\overline{\mathcal{V}}^{(2)}$, which are very simply related to each other:

$$\begin{aligned} &\mathcal{E}^{(0)} = \mathcal{K}^{(0)} = K^{(0)} A \quad , && \mathcal{E}^{(1)} = \overline{\mathcal{V}}^{(1)} \\ &\mathcal{E}^{(2)} = \mathcal{K}^{(2)} + \overline{\mathcal{V}}^{(2)} \quad , \quad \text{with} && \overline{\mathcal{V}}^{(2)} = -2\,\mathcal{K}^{(2)}. \end{aligned} \tag{2}$$

In the ground state of the system, the second order correction

$$\mathcal{E}^{(2)} = -\mathcal{K}^{(2)} = \tfrac{1}{2}\overline{\mathcal{V}}^{(2)} \tag{3}$$

is always negative, i.e. is equivalent to a tightening of the binding.

In order, therefore, to obtain the energy $\mathcal{E}$ up to the second approximation, it is sufficient to calculate the potential energy up to that approximation, and since the potential energy is itself of the first order, it suffices to use the first order approximation for the wave-function of the ground state. The latter can be written in the form of an expansion

$$\Psi = \frac{1}{\sqrt{1+\Sigma\,|a_\mu|^2}}\,(\Psi_0 + \sum_\mu a_\mu\,\Psi_\mu) \tag{4}$$

in terms of the Slater-determinants Ψ_0, Ψ_μ, corresponding to the initial approximation for ground state and excited configurations. The only excited states Ψ_μ which combine with the ground state are those in which two individual states, say $n_i^{(1)}$, $n_i^{(2)}$, occupied in the ground state configuration, are replaced by two others, $n_f^{(1)}$, $n_f^{(2)}$, unoccupied in that configuration. The corresponding expansion coefficient a_μ is, in first approximation, for a nuclear potential of the general form (*8.2*–1),

$$a_\mu = \frac{(n_i^{(1)}\,n_i^{(2)}\,|\,\mathcal{O}\,\mathcal{J}(r)\,|\,n_f^{(1)}\,n_f^{(2)}) - (n_i^{(1)}\,n_i^{(2)}\,|\,\mathcal{O}\,\mathcal{J}(r)\,|\,n_f^{(2)}\,n_f^{(1)})}{\mathsf{E}_i^{(1)} + \mathsf{E}_i^{(2)} - (\mathsf{E}_f^{(1)} + \mathsf{E}_f^{(2)})}; \tag{5}$$

$\mathsf{E} = p^2/2M$ denotes the energy of the individual state of momentum $\vec{p}$. Using the abbreviations i, f for the two sets of values of the spin and isotopic coordinates $\sigma_z^{(1)\prime}$, $\sigma_z^{(2)\prime}$, $\tau_3^{(1)\prime}$, $\tau_3^{(2)\prime}$, this may be written, more explicitly,

$$(i\,|\,a_\mu\,|\,f) = \frac{(i\,|\,\mathcal{O}\,|\,f)\,(\vec{p}_i^{(1)}\,\vec{p}_i^{(2)}\,|\,\mathcal{J}(r)\,|\,\vec{p}_f^{(1)}\,\vec{p}_f^{(2)}) - (i\,|\,\mathcal{O}P_\sigma P_\tau\,|\,f)\,(\vec{p}_i^{(1)}\,\vec{p}_i^{(2)}\,|\,\mathcal{J}(r)\,|\,\vec{p}_f^{(2)}\,\vec{p}_f^{(1)})}{\mathsf{E}_i^{(1)} + \mathsf{E}_i^{(2)} - (\mathsf{E}_f^{(1)} + \mathsf{E}_f^{(2)})}. \tag{6}$$

The expectation value of any operator (such as the nuclear interaction energy) of the form

$$\mathcal{W} = \tfrac{1}{2}\sum_{i,k} \mathcal{W}^{(ik)}(Q^{(i)},Q^{(k)}) \tag{7}$$

in the state (4) can be written (this particular form will presently be useful for another purpose)

$$\begin{aligned}\text{av}\, \mathcal{W} &= \tfrac{1}{2} A(A-1) \int_{Q^{(1)}} \int_{Q^{(2)}} g(Q^{(1)},Q^{(2)})\, \mathcal{W}^{(12)}(Q^{(1)},Q^{(2)}) \\ g(Q^{(1)},Q^{(2)}) &= \int_{Q^{(3)}} \cdots \int_{Q^{(A)}} \Psi^*(Q^{(1)},\ldots, Q^{(A)})\, \Psi(Q^{(1)},\ldots, Q^{(A)}),\end{aligned} \tag{8}$$

or, using again the same notation i, f for the spin and isotopic coordinates,

$$\text{av}\, \mathcal{W} = \tfrac{1}{2} A(A-1) \int dv^{(1)} dv^{(2)} \sum_{i,f} (i \,|\, g(\vec{x}^{(1)},\vec{x}^{(2)}) \,|\, f)\, (f \,|\, \mathcal{W}^{(12)}(\vec{x}^{(1)},\vec{x}^{(2)}) \,|\, i) \tag{9}$$

$$(i \,|\, g(\vec{x}^{(1)},\vec{x}^{(2)}) \,|\, f) = \int_{Q^{(3)}} \cdots \int_{Q^{(A)}} \Psi^*(\vec{x}^{(1)},\vec{x}^{(2)}; i; Q^{(3)},\ldots, Q^{(A)})\, \Psi(\vec{x}^{(1)},\vec{x}^{(2)}; f; Q^{(3)},\ldots, Q^{(A)}).$$

For reasons which will soon be apparent, we shall call g the *correlation operator*. In the state (4), its two first approximations are

$$\begin{aligned}(i \,|\, g \,|\, f)^{(0)} &= \frac{1}{A(A-1)} \sum_{\vec{p}^{(1)} \vec{p}^{(2)}}^{(\text{occ})} \varphi^*_{\vec{p}^{(1)}}(\vec{x}^{(1)})\, \varphi^*_{\vec{p}^{(2)}}(\vec{x}^{(2)})\, [(i \,|\, 1 \,|\, f)\, \varphi_{\vec{p}^{(1)}}(\vec{x}^{(1)})\, \varphi_{\vec{p}^{(2)}}(\vec{x}^{(2)}) \\ &\qquad - (i \,|\, P_\sigma P_\tau \,|\, f)\, \varphi_{\vec{p}^{(2)}}(\vec{x}^{(1)})\, \varphi_{\vec{p}^{(1)}}(\vec{x}^{(2)})] \\ (i \,|\, g \,|\, f)^{(1)} &= \\ \frac{1}{A(A-1)} \cdot 2\Re &\sum_{\vec{p}^{(1)}_i \vec{p}^{(2)}_i}^{(\text{occ})} \sum_{\vec{p}^{(1)}_f \vec{p}^{(2)}_f}^{(\text{unocc})} \varphi^*_{\vec{p}^{(1)}_i}(\vec{x}^{(1)})\, \varphi^*_{\vec{p}^{(2)}_i}(\vec{x}^{(2)})\, \varphi_{\vec{p}^{(1)}_f}(\vec{x}^{(1)})\, \varphi_{\vec{p}^{(2)}_f}(\vec{x}^{(2)})\, (i \,|\, a_\mu \,|\, f)\,;\end{aligned} \tag{10}$$

in these formulae, the summations are to be extended, according to the indication $^{(\text{occ})}$ or $^{(\text{unocc})}$, over all occupied or all unoccupied individual states of momentum $\vec{p}$ in the configuration Ψ_0, each state being counted only once (such summations can be replaced, as usual, by integrations over the corresponding domains of phase space).

From the formulae (9) and (10) we derive for the total energy the well-known expressions

$$\begin{aligned}\mathcal{E}^{(1)} = \bar{\mathcal{V}}^{(1)} &= \tfrac{1}{2} \sum_{\vec{p}^{(1)} \vec{p}^{(2)}}^{(\text{occ})} [(\vec{p}^{(1)} \vec{p}^{(2)} \,|\, J(r) \,|\, \vec{p}^{(1)} \vec{p}^{(2)})\, \text{tr}\, \mathcal{O} \\ &\qquad - (\vec{p}^{(1)} \vec{p}^{(2)} \,|\, J(r) \,|\, \vec{p}^{(2)} \vec{p}^{(1)})\, \text{tr}\, \mathcal{O} P_\sigma P_\tau] \\ \mathcal{E}^{(2)} = \tfrac{1}{2} \bar{\mathcal{V}}^{(2)} &= -\tfrac{1}{2} \sum_{\vec{p}^{(1)}_i \vec{p}^{(2)}_i}^{(\text{occ})} \sum_{\vec{p}^{(1)}_f \vec{p}^{(2)}_f}^{(\text{unocc})} \frac{(\vec{p}^{(1)}_i \vec{p}^{(2)}_i \,|\, \mathfrak{K} \,|\, \vec{p}^{(1)}_f \vec{p}^{(2)}_f)}{\mathrm{E}^{(1)}_f + \mathrm{E}^{(2)}_f - (\mathrm{E}^{(1)}_i + \mathrm{E}^{(2)}_i)},\end{aligned} \tag{11}$$

with the abbreviation

$$\begin{aligned}(\vec{p}^{(1)}_i \vec{p}^{(2)}_i \,|\, \mathfrak{K} \,|\, \vec{p}^{(1)}_f \vec{p}^{(2)}_f) &= |(\vec{p}^{(1)}_i \vec{p}^{(2)}_i \,|\, J(r) \,|\, \vec{p}^{(1)}_f \vec{p}^{(2)}_f)|^2\, \text{tr}\, \mathcal{O}^2 \\ &- (\vec{p}^{(1)}_i \vec{p}^{(2)}_i \,|\, J(r) \,|\, \vec{p}^{(1)}_f \vec{p}^{(2)}_f)\, (\vec{p}^{(1)}_i \vec{p}^{(2)}_i \,|\, J(r) \,|\, \vec{p}^{(2)}_f \vec{p}^{(1)}_f)^*\, \text{tr}\, \mathcal{O}^2 P_\sigma P_\tau.\end{aligned} \tag{12}$$

In $\mathcal{E}^{(1)}$ as well as $\mathcal{E}^{(2)}$, there occur an ordinary and an exchange term; the formula for $\mathcal{E}^{(1)}$ is actually the same as (*11.11*–24), in which the coefficients $\mathrm{tr}\,O$ and $\mathrm{tr}\,OP_\sigma P_\tau$, as given by (*8.21*–13), had been deduced by a more elaborate analysis. Let us use again the notations (*12.13*–8, 10) and express all momenta in terms of $p_m^{(0)}$ by

$$\vec{u} = \frac{\vec{p}}{p_m^{(0)}} \tag{13}$$

or

$$\frac{\vec{p}}{\hbar\varkappa} = \vec{u}\,\mathrm{r}. \tag{14}$$

The cofactor $w_p(p_0)$ occurring in the expression (*12.11*–4) for the matrix-elements of the nuclear potential then takes, according to (*12.11*–3) and (14), the form

$$w_p(p_0) = \frac{1}{A}\cdot\frac{4}{3\pi}\,\mathrm{r}^3\,F(u_0\mathrm{r}). \tag{15}$$

The formulae (11) accordingly reduce to

$$\begin{aligned}
\mathcal{E}^{(1)} &= \frac{F(0)}{4\pi}\,A\,J\,[\mathrm{tr}\,O\cdot\frac{\mathrm{r}^3}{6} - \mathrm{tr}\,O\,P_\sigma\,P_\tau\cdot f^{(1)}_{\mathrm{exch}}(\mathrm{r})] \\
f^{(1)}_{\mathrm{exch}}(\mathrm{r}) &= \frac{\mathrm{r}^3}{6}\left(\frac{3}{4\pi}\right)^2\frac{1}{F(0)}\int dv_u\int dv_{u'}\,F(\mathrm{r}\,|\vec{u}-\vec{u}'|)
\end{aligned} \tag{16}$$

(both integrals to be taken independently over the spheres $u \leq 1$, $u' \leq 1$),

$$\begin{aligned}
\mathcal{E}^{(2)} &= -\frac{1}{320\pi}\,A\,J^2\,\frac{M}{\hbar^2\varkappa^2}\,[\mathrm{tr}\,O^2\cdot f^{(2)}_{\mathrm{ord}}(\mathrm{r}) - \mathrm{tr}\,O^2\,P_\sigma\,P_\tau\cdot f^{(2)}_{\mathrm{exch}}(\mathrm{r})] \\
f^{(2)}_{\mathrm{ord}}(\mathrm{r}) &= \frac{15\,\mathrm{r}^4}{8\pi^4}\int dv_u\,dv_{u'}\,dv_{u''}\,\frac{[F(\mathrm{r}u')]^2}{\vec{u}'(\vec{u}'+\vec{u}''-\vec{u})} \\
f^{(2)}_{\mathrm{exch}}(\mathrm{r}) &= \frac{15\,\mathrm{r}^4}{8\pi^4}\int dv_u\,dv_{u'}\,dv_{u''}\,\frac{F(\mathrm{r}u')F(\mathrm{r}\,|\vec{u}'+\vec{u}''-\vec{u}|)}{\vec{u}'(\vec{u}'+\vec{u}''-\vec{u})}
\end{aligned} \tag{17}$$

(domain of integration defined by the inequalities *

$u \leq 1$, $u'' \leq 1$, $|\vec{u}-\vec{u}'| \geq 1$, $|\vec{u}'+\vec{u}''| \geq 1$).

In this form they have been derived by EULER [37] for the special case of the Gauss potential **.

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* We have put $\vec{u} = \vec{p}_i^{(1)}/p_m^{(0)}$, $\vec{u}' = (\vec{p}_i^{(1)} - \vec{p}_f^{(1)})/p_m^{(0)}$, $\vec{u}'' = \vec{p}_i^{(2)}/p_m^{(0)}$.

** A factor 5 has been introduced in our definition of the functions $f^{(2)}_{\mathrm{ord}}$ and $f^{(2)}_{\mathrm{exch}}$ only in order to conform with Euler's definitions.

Volz' saturation condition (*11.31*–2) implies, by (*8.21*–13), tr $\mathcal{O} = 0$ and so causes the non-saturation term in $\mathcal{E}^{(1)}$ to disappear; moreover, it gives, on account of (*8.21*–13) and (*8.2*–12),

$$\mathrm{tr}\, \mathcal{O}\, P_\sigma P_\tau = 6\,(1+q) \approx 9{,}6\,, \tag{18}$$

thus completely fixing the constant factor in $\mathcal{E}^{(1)}$. At small densities of the nuclear matter, the factor $f^{(1)}_{\mathrm{exch}}$ becomes $\approx \mathrm{r}^3/6$, i.e. increases proportionally to this density. But for large densities, it tends asymptotically towards a constant value. This behaviour, which is in conformity with the saturation property of nuclear binding, is conditioned by the fact that, according to the definition (*12.11*–3) of $F(u)$, the main contributions to the energy come from transitions in which the transfer of momentum p_0 is smaller than $\hbar\varkappa$, i.e. $p_0/p^{(0)}_m \lesssim \mathrm{r}^{-1}$. At small densities, the occupied states are few and their momenta therefore not widely different, so that the required transitions are possible from most of them; at high densities, on the contrary, only a small fraction of the nucleons can take part in such transitions: any nucleon can exchange its momentum with only $(p_0/p^{(0)}_m)^3 \approx \mathrm{r}^{-3}$ of the others, which just compensates the increase ($\sim \mathrm{r}^3$) of the binding energy due to the diminution of the average distance between the nucleons. The general properties of $f^{(1)}_{\mathrm{exch}}$ are clearly illustrated by the explicit expression which can be obtained for a nuclear potential of the Gauss type. This corresponds to

$$F(u) = \tfrac{1}{2}\sqrt{\pi}\, e^{-u^2/4}\,. \tag{19}$$

The evaluation of $f^{(1)}_{\mathrm{exch}}$ is then elementary and yields

$$f^{(1)}_{\mathrm{exch}}(\mathrm{r}) = 2\int_0^{\mathrm{r}} e^{-t^2}\,dt + \frac{1}{\mathrm{r}^3}\,[2 - 3\,\mathrm{r}^2 + (\mathrm{r}^2 - 2)\,e^{-\mathrm{r}^2}]\,; \tag{20}$$

one has $f^{(1)}_{\mathrm{exch}}(\infty) = \sqrt{\pi}$.

Going over to $\mathcal{E}^{(2)}$, we first determine the coefficients tr$\mathcal{O}^2$ and tr$\mathcal{O}^2 P_\sigma P_\tau$ with the help of the formulae (*8.21*–16) and (*11.33*–12, 13), the latter depending on both saturation conditions (*11.33*–7); we get

$$\begin{aligned} \mathrm{tr}\, \mathcal{O}^2 \qquad &= 4\,(1 + 3\,q^2) = 4\cdot 2{,}153 \\ \mathrm{tr}\, \mathcal{O}^2\, P_\sigma P_\tau &= 2\,(-1 + 3\,q^2) = 2\cdot 0{,}153. \end{aligned} \tag{21}$$

The numerical values correspond to $q = 0{,}62$, the value which, according to table *6.432*, fits with the two-nucleon data in the case of the Gauss potential. This means that $\mathcal{E}^{(2)}$ depends on the combination

$$f^{(2)}_{\mathrm{ord}} - 0{,}036\, f^{(2)}_{\mathrm{exch}}\,, \tag{22}$$

so that the exchange contribution is of little significance.

It has been observed by WATANABE [39] that a very slight modification of Kemmer's saturation condition $a_\sigma = 0$ (while maintaining $a_0 = 0$) would suffice to get rid entirely of second order exchange effects. In fact, one can make tr$\mathcal{O}^2 P_\sigma P_\tau$ exactly zero, i.e. satisfy the condition

$$9\cdot {}^3p^2 + {}^1p^2 = 3\,(1+q^2) \tag{23}$$

by

$$a_\sigma = \tfrac{1}{40}\,[9q-1-\sqrt{3\,(7q^2-6q+7)}] = 0{,}0087\,, \tag{24}$$

as is easily found by inserting in (23) the values of ${}^3p$ and ${}^1p$ in terms of a_σ and q deduced from (8.2–9, 12), with $a_0 = 0$. Moreover, the condition (23) implies only a slight change of tr $\mathcal{O}^2$:

$$\tfrac{1}{4}\,\mathrm{tr}\,\mathcal{O}^2 = \tfrac{3}{2}\,(1+q^2) = 2{,}076. \tag{25}$$

The discussion of the $f^{(2)}$'s has only been performed for the case of the Gauss potential. Their calculation requires a lengthy procedure of approximation, for the details of which the reader is referred to Euler's paper. The behaviour of $f^{(2)}_{\mathrm{ord}}$ is found to be quite similar to that of $f^{(1)}_{\mathrm{exch}}$:

$$\begin{aligned} f^{(2)}_{\mathrm{ord}}(\mathrm{r}) &\approx \tfrac{5}{3}\sqrt{2\pi}\,\mathrm{r}^3 \qquad [\text{for } \mathrm{r} \ll 1], \\ f^{(2)}_{\mathrm{ord}}(\mathrm{r}) &\simeq 10\,(1-\log 2) - \tfrac{5}{2}\,\mathrm{r}^{-2} + \tfrac{1}{2}\,\mathrm{r}^{-4} + 0{,}16\,\mathrm{r}^{-6} \\ &\qquad [\text{for } \mathrm{r} \gtrsim 1{,}6,\ \text{with an error} \lesssim 2\%]. \end{aligned} \tag{26}$$

The saturation feature is again due to the limitation of the possible transitions resulting from the short range of the nuclear force and the exclusion principle. As to $f^{(2)}_{\mathrm{exch}}(\mathrm{r})$, it is the same as $f^{(2)}_{\mathrm{ord}}$ at small r: then it becomes smaller than $f^{(2)}_{\mathrm{ord}}$, reaches a maximum and tends asymptotically to zero for $\mathrm{r} \to \infty$. It can further be verified that the higher approximations of the energy all show the same behaviour as $f^{(2)}_{\mathrm{exch}}(\mathrm{r})$.

Summing up, we are now in a position to write down the complete expression for the volume energy per nucleon in explicit form for a Gauss potential. It will be convenient to revert to our notation (5.11–9), involving the quantity $\hbar^2\varkappa^2/M$ as unit of energy. Remembering that the mean kinetic energy per nucleon is just $K^{(0)} = 0{,}3\ (\hbar^2\varkappa^2/M)\,\mathrm{r}^2$, we have *, by (16), (17), (18), (21), (22),

$$\begin{aligned} -\eta_1 &= -\frac{M}{\hbar^2\varkappa^2}\,\varepsilon_1 \\ &= 0{,}3\,\mathrm{r}^2 - 0{,}685\,b\,f^{(1)}_{\mathrm{exch}}(\mathrm{r}) - 0{,}00857\,b^2\,[f^{(2)}_{\mathrm{ord}}(\mathrm{r}) - 0{,}036\,f^{(2)}_{\mathrm{exch}}(\mathrm{r})]: \end{aligned} \tag{27}$$

this is (apart from the numerical values of the coefficients) the formula established by Euler; $f^{(1)}_{\mathrm{exch}}$ and $f^{(2)}_{\mathrm{ord}}$ are given by (20) and (26), respectively.

For a given nuclear potential, i.e. for given b, the best values of η_1 and r (and consequently of ε_1 and r_0, $\varkappa$ being also given) are determined by (27) together with the condition of stationary energy

$$\frac{d\eta_1}{d\mathrm{r}} = 0. \tag{28}$$

* With Watanabe's modification (23), (25), the last term would be

$$-0{,}00826\,b^2\,f^{(2)}_{\mathrm{ord}}(\mathrm{r}).$$

The data resulting from the analysis of the two-nucleon system (table *6.432*),

$$\varkappa^{-1} = 1{,}9 \cdot 10^{-13}\,\text{cm} \quad , \quad J = 43{,}7\ \text{MeV}, \tag{29}$$

correspond to $\hbar^2\varkappa^2/M = 11{,}5$ MeV, and therefore

$$b = 3{,}80. \tag{30}$$

However, if we adopt this value for b, and draw the graph $\eta_1(\mathrm{r})$, we find for the extreme

$$\begin{aligned} \mathrm{r}_0 &\approx 1{,}84 \quad , \quad \eta_1 \approx 0{,}464 \\ \text{or} \qquad r_0 &\approx 0{,}56\, d \quad , \quad \varepsilon_1 \approx 5{,}3\ \text{MeV}, \end{aligned} \tag{31}$$

i.e. too large a nucleon radius r_0 and, above all, much too small an average binding energy. The relative contributions of $\mathcal{E}^{(0)}$, $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ to this result * are, in units $(\hbar^2\varkappa^2/M) \cdot A$,

$$\mathcal{E}^{(0)} = 0{,}97, \qquad \mathcal{E}^{(1)} = -1{,}14, \qquad \mathcal{E}^{(2)} = -0{,}29: \tag{32}$$

Such figures illustrate at the same time the considerable improvement effected by the second approximation (without it, we would get a maximum value of $\varepsilon_1 \approx 1{,}95$ MeV) and the very poor convergence of the method.

Conversely, we may try to determine the nuclear potential for which Euler's formula would be in agreement with the empirical values of r_0 and ε_1. For this purpose, it is more convenient to take as energy unit the maximum zero-point energy $\mathrm{E}_m^{(0)}$, which depends only on r_0; we put accordingly, taking account of (*12.13*–10),

$$\eta_1' = \frac{\varepsilon_1}{\mathrm{E}_m^{(0)}} = \frac{2}{\mathrm{r}^2}\,\eta_1 \quad , \quad b' = \frac{J}{\mathrm{E}_m^{(0)}} = \frac{2}{\mathrm{r}^2}\, b \tag{33}$$

and rewrite Euler's formula and the condition of stationary energy in the form

$$\begin{aligned} \eta_1' &= -0{,}6 + 0{,}685\, b'\, f_{\text{exch}}^{(1)}(\mathrm{r}_0) + 0{,}00428\, b'^2\, \mathrm{r}_0^2\, [f_{\text{ord}}^{(2)} - 0{,}036\, f_{\text{exch}}^{(2)}] \\ 0 &= -1{,}2 + 0{,}685\, b'\, \mathrm{r}_0 \frac{d}{d\mathrm{r}_0} f_{\text{exch}}^{(1)} + 0{,}00428\, b'^2\, \mathrm{r}_0^3 \frac{d}{d\mathrm{r}_0} [f_{\text{ord}}^{(2)} - 0{,}036\, f_{\text{exch}}^{(2)}]. \end{aligned} \tag{34}$$

The second equation (34) allows us to determine b' in terms of r_0, the first one subsequently to calculate the corresponding η_1'. The empirical value $\eta_1' = 0{,}60$ is found to correspond roughly to

$$\mathrm{r}_0 \approx 2{,}6 \quad , \quad b' \approx 2{,}05 \quad (b \approx 6{,}93) \tag{35}$$

i.e.

$$\varkappa^{-1} \approx 2{,}4 \cdot 10^{-13}\,\text{cm}, \quad J \approx 49{,}6\ \text{MeV}: \tag{36}$$

both range and strength of potential ought thus to be chosen appreciably larger than is indicated by the experiments on two-nucleon systems. As a

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* More exactly, we quote the values for $\mathrm{r} = 1{,}8$.

kind of compromise, one might adopt for the purpose of numerical estimates, the set

$$\mathrm{r}_0 = 2{,}5 \quad , \quad b = 4{,}2 \tag{37}$$

used by WATANABE [39]. Assuming $r_0 = \frac{1}{2}d$, the value $\mathrm{r}_0 = 2{,}5$ corresponds to a range $\varkappa^{-1} = 2{,}33 \cdot 10^{-13}$ cm. The value $b = 4{,}2$ is that which, for such a range, would yield about the right binding energy for the deuteron *. But, of course, this range is much too large to be compatible with the scattering experiments, and the average binding energy ε_1 comes out much too low ($\approx 4{,}1$ MeV).

From (34) we may derive an explicit formula for the velocity of sound in nuclear matter, as defined by (*12.13*–16):

$$C^2 = \frac{2\,\mathrm{E}_m}{9M} \Big\{ 0{,}6 + 2{,}055\, b'\, \mathrm{r}_0^{-3} [\mathrm{r}_0^2 - 4 + e^{-\mathrm{r}_0^2} (\mathrm{r}_0^4 + 3\,\mathrm{r}_0^2 + 4)] + 0{,}00214\, b'^2 \left[15 + \frac{4{,}4}{\mathrm{r}_0^2} - \frac{30{,}91}{\mathrm{r}_0^4} + \ldots \right] \Big\}. \tag{38}$$

With the set of parameters (35), this gives

$$C^2 \approx 8{,}1 \cdot 10^{-3}. \tag{39}$$

12.22. *The momentum distribution.* In the initial approximation, corresponding to the absolute zero of temperature for the degenerate Fermi gas of nucleons, the distribution of the momenta of these nucleons is simply described by saying that all the lowest states are occupied up to a limit fixed by the total number of nucleons. The effect of the nuclear interactions, by causing transitions from these states into unoccupied ones, will be to spread out this distribution in the neighbourhood of the limiting momentum $p_m^{(0)}$, lowering the density in momentum space below $p_m^{(0)}$ and giving rise to a "tail" in the distribution above this value. A quantitative study of this effect is due to WATANABE [39].

We again restrict ourselves to the consideration of a standard heavy nucleus, in the stationary state described by the formula (4) above. Let $N_\mu(n)$ denote the number (0 or 1) of nucleons occupying, in the configuration μ, any one of the four substates of momentum $\vec{p}$, characterized by an index n which stands for the spin and isotopic coordinates. The probability $F(\vec{p}) \cdot 4V(2\pi\hbar)^{-3}\,dv_p$ for a nucleon to have its momentum in the interval $(\vec{p}, \vec{p} + d\vec{p})$ is then clearly defined by

$$F(\vec{p}) = \frac{1}{A} \cdot \frac{\mathrm{av}_n\, [\varepsilon + \sum_\mu |a_\mu|^2\, N_\mu(n)]}{1 + \sum_\mu |a_\mu|^2} \tag{40}$$

with

$$\varepsilon = 1 \quad \text{if } p < p_m^{(0)} \quad , \quad \varepsilon = 0 \quad \text{if } p > p_m^{(0)}.$$

* From a table (B & B, table III, p. 111) calculated for a deuteron binding energy of 2,15 MeV, it can indeed be inferred that when the range of the Gauss potential increases from 1,9 to 2,3 (10^{-13} cm), the b-value will become about 10 % larger.

the average having to be taken over the substates n. Assuming

$$\sum_{\mu} |a_\mu|^2 \ll 1, \tag{41}$$

a condition necessary for the rapid convergence of the approximation procedure embodied in formula (4), we may write (40) more simply

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} \{1 - \mathrm{av}_n \sum_{\mu} |a_\mu|^2 [1 - N_\mu(n)]\} \qquad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \mathrm{av}_n \sum_{\mu} |a_\mu|^2 N_\mu(n) \qquad (p > p_m^{(0)}), \end{aligned} \tag{42}$$

or, according to (6),

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} \left\{ 1 - \tfrac{1}{4} \sum_{\vec{p}_i^{(2)}}^{(\mathrm{occ})} \sum_{\vec{p}_f^{(1)} \vec{p}_f^{(2)}}^{(\mathrm{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \mathfrak{K} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{[\mathrm{E}_i^{(1)} + \mathrm{E}_i^{(2)} - (\mathrm{E}_f^{(1)} + \mathrm{E}_f^{(2)})]^2} \right\}_{\vec{p}_i^{(1)} = \vec{p}} \qquad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \left\{ \tfrac{1}{4} \sum_{\vec{p}_i^{(1)} \vec{p}_i^{(2)}}^{(\mathrm{occ})} \sum_{\vec{p}_f^{(2)}}^{(\mathrm{unocc})} \frac{(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \mathfrak{K} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})}{[\mathrm{E}_i^{(1)} + \mathrm{E}_i^{(2)} - (\mathrm{E}_f^{(1)} + \mathrm{E}_f^{(2)})]^2} \right\}_{\vec{p}_f^{(1)} = \vec{p}} \qquad (p > p_m^{(0)}). \end{aligned} \tag{43}$$

In these formulae use has been made of the notation (12); the factors $\frac{1}{4}$ account for the fact that the indicated summations over momenta are to be carried out independently of each other, whereas each pair of momenta $\vec{p}_i^{(1)}$, $\vec{p}_i^{(2)}$ or $\vec{p}_f^{(1)}$, $\vec{p}_f^{(2)}$ ought to be counted only once when performing the summation over the configurations μ in (42). The traces occurring in the numerators $(\vec{p}_i^{(1)} \vec{p}_i^{(2)} | \mathfrak{K} | \vec{p}_f^{(1)} \vec{p}_f^{(2)})$ given by (12) arise from the averaging over the spin and charge substates; as above, we shall neglect the exchange term, with the coefficient $\mathrm{tr}\, \mathcal{O}^2 P_\sigma P_\tau$. Performing again the variable substitution (13) or (14), with the resulting formula (15), we finally put the expressions (43) into the form

$$\begin{aligned} F(\vec{p}) &= \frac{1}{A} [1 - \beta \varphi_{\mathrm{occ}}(u;\mathrm{r})] \qquad (p < p_m^{(0)}) \\ F(\vec{p}) &= \frac{1}{A} \cdot \beta \varphi_{\mathrm{unocc}}(u;\mathrm{r}) \qquad (p > p_m^{(0)}), \end{aligned} \tag{44}$$

with

$$\beta = \tfrac{1}{4}\, \mathrm{tr}\, \mathcal{O}^2 \cdot \frac{b^2 \mathrm{r}^2}{16 \pi^4} [F(0)]^2 \tag{45}$$

and

$$\varphi(u;\mathrm{r}) = \int dv_{u'} \int dv_{u''} \frac{[F(\mathrm{r}u')/F(0)]^2}{[\vec{u}' (\vec{u}' + \vec{u}'' - \vec{u})]^2}, \tag{46}$$

the domains of integration being defined by the inequalities

$$u'' \leqslant 1 \quad , \quad |\vec{u}' + \vec{u}''| \geqslant 1$$

and

$$\begin{cases} |\vec{u} - \vec{u}'| \geqslant 1 & \text{for } \varphi_{\text{occ}} \\ |\vec{u} + \vec{u}'| \leqslant 1 & \text{for } \varphi_{\text{unocc}}. \end{cases} \tag{47}$$

The evaluation of these intricate integrals, carried out by Watanabe in the case of a Gauss potential, yields the expected results: he finds that, for any fixed value of r, φ_{occ} increases when u increases from 0 to 1, while for u increasing further from 1 upwards, φ_{unocc} decreases rather steeply, tending to zero roughly as $e^{-r^2 u^2/2}$ for $u \to \infty$. For $u = 1$, there is in general a discontinuity of $F(\vec{p})$; e.g. for the set of parameters (37), one has

$$\varphi_{\text{occ}}(0) = 0{,}19 \quad , \quad \varphi_{\text{occ}}(1) = 4{,}51 \quad , \quad \varphi_{\text{unocc}}(1) = 3{,}90, \tag{48}$$

while according to (45), (19) and (25) $\beta = 0{,}116$: the corresponding distribution function $F(\vec{p})$ is represented in fig. *12.22*.

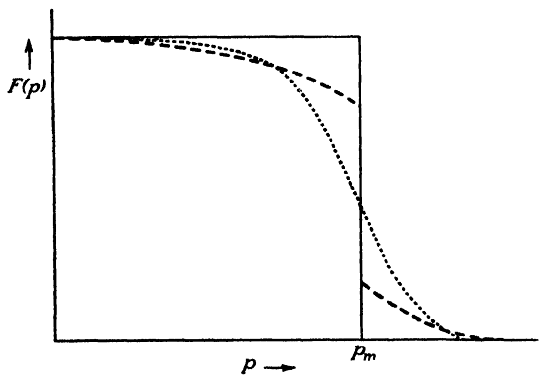

Fig. 12.22. Momentum distribution in standard heavy nucleus; ——— zero-point Fermi distribution, - - - - - distribution due to nuclear interaction, ... Fermi distribution at „equivalent" temperature.

By allowing us to calculate a more accurate value of the kinetic energy, the determination of $F(\vec{p})$ affords an alternative method of estimating the second order approximation to the nuclear energy. In fact, we get

$$\begin{aligned} K/A &= \int_0^\infty \frac{p^2}{2M} F(\vec{p}) \frac{4V}{(2\pi\hbar)^3} dv_p \\ &= K\{1 + 5\beta [\int_1^\infty u^4 du\, \varphi_{\text{unocc}}(u;r_0) - \int_0^1 u^4 du\, \varphi_{\text{occ}}(u;r_0)]\}; \end{aligned} \tag{49}$$

the correction term is just $K^{(2)}/A$, i.e., by (3), $-\mathcal{E}^{(2)}/A$. For the parameter values (37), Watanabe gives

$$K^{(2)}/K^{(0)} = 2{,}95\beta = 0{,}34, \tag{50}$$

but there is a serious discrepancy between this figure and that ($\approx 0{,}21$)

resulting from (27) for the same choice of parameters. As, however, all numerical data of this section can have only an illustrative significance, we shall not try to remove this discrepancy.

The modification of the momentum distribution brought about by the nuclear interactions is qualitatively of the same kind as that due to thermal agitation. It is interesting to compare the distribution (44) with that pertaining to an *ideal* Fermi gas (without nuclear interactions), the temperature of which is chosen so as to correspond to the same average kinetic energy (49). On account of the formulae (*9.42*–8, 10) and (50) with (*9.41*–4), this condition fixes the "equivalent" temperature T_0 by

$$\frac{5\pi^2}{12}\left(\frac{T_0}{\mathsf{E}_m}\right)^2 = 2{,}95\,\beta = 0{,}34, \tag{51}$$

whence

$$T_0 = 0{,}29\ \mathsf{E}_m \approx 7\ \mathrm{MeV} \tag{52}$$

by (*9.41*–5). It appears from fig. *12.22* that the graphs of the two momentum distributions offer indeed a striking similarity.

We can finally estimate in how far the convergence criterion (41) is fulfilled. We have

$$\begin{aligned}\sum_\mu |a_\mu|^2 &= \tfrac{1}{2} \sum_{\vec{p}_i^{(1)}\vec{p}_i^{(2)}}^{(\mathrm{occ})} \sum_{\vec{p}_f^{(1)}\vec{p}_f^{(2)}}^{(\mathrm{unocc})} \frac{(\vec{p}_i^{(1)}\vec{p}_i^{(2)}|\mathfrak{K}|\vec{p}_f^{(1)}\vec{p}_f^{(2)})}{[\mathsf{E}_i^{(1)}+\mathsf{E}_i^{(2)}-(\mathsf{E}_f^{(1)}+\mathsf{E}_f^{(2)})]^2} \\ &= \tfrac{3}{2}\, A\,\beta \int_0^1 u^2\,du\ \varphi_{\mathrm{occ}}(u;\mathrm{r}_0) = \tfrac{3}{2}\, A\,\beta \int_1^\infty u^2\,du\ \varphi_{\mathrm{unocc}}(u;\mathrm{r}_0),\end{aligned} \tag{53}$$

which yields, according to Watanabe,

$$\sum_\mu |a_\mu|^2 = 1{,}18\,A\beta = 0{,}137\,A \tag{54}$$

for the parameter values (37). The convergence is thus quite bad, even for small integral values of A, and gets worse for heavier nuclei: the individual models become in fact more and more inadequate as the number of nucleons increases, and the saturation properties of the nuclear interactions come more and more to the foreground. Still, as observed by Watanabe, the relation (54) shows that results derived by the perturbation method on the Fermi gas model should *formally* be valid for $A \ll 1$, and this remark may possibly lead to less pessimistic conclusions with respect to certain quantities depending in some simple way on the mass number. Thus, it follows from the fact that the probability $F(\vec{p})\cdot 4V(2\pi\hbar)^{-3}\cdot dv_p$, according to its definition (40), is independent of A, that the same property holds for the expectation value of any quantity (e.g. the kinetic energy) additive with respect to the different nucleons : we would therefore expect the pertubation method to yield a fair approximation for such expectation values.

In particular, this conclusion should apply to the average binding energy per nucleon, which, as we have seen, is also independent of A, in conformity with the saturation character of the nuclear forces. If in spite of that no satisfactory quantitative agreement with the empirical data can be reached, it is because the model does not allow sufficiently for the markedly local character of the interaction of any particular nucleon with the others. It must be emphasized, however, that even Euler's approximation means a considerable step towards the due recognition of the importance of local fluctuations in the distribution of the nucleons, arising from their mutual interaction: this will appear from a closer discussion of the spatial correlations between the individual nucleons, which are found in this approximation.

12.23. *The spatial correlations.* In order to study the spatial distribution of the nucleons, we have to turn back to the correlation operator introduced in formula (8) or (9): in fact, the diagonal matrix-elements of this operator (which is obviously spherically symmetrical) give the probability for two nucleons of specified charge and spin to be at a distance $r = |\vec{x}| = |\vec{x}^{(1)} - \vec{x}^{(2)}|$ apart. We shall distinguish two cases, according as the nucleons considered have or have not the same charge and spin orientation, i.e. do or do not belong to the same isotopic and spin substates. Two nucleons with the same charge and spin we will call *congruent* *; if they differ either in charge or spin or both, they will be called *non-congruent*. The *spatial correlation* of a pair of congruent or non-congruent nucleons will then be measured by the total probability

$$g(r) = \sum_i (i\,|\,g\,|\,i), \tag{55}$$

the sum extending over the spin and isotopic states which correspond to congruence or non-congruence of the nucleon pair: congruent nucleons can be in any of the four states $^3(\tau)_{\pm 1}\,{}^3(\sigma)_{\pm 1}$, non-congruent nucleons in any of the other 12 possible states. Quantities pertaining to a pair of congruent or non-congruent nucleons will be distinguished by the respective suffixes $=$, $\neq$.

In terms of the ordinary and mixed densities defined by (*11.11*–12, 13, 23), the initial approximation to the correlation operator, given by the first formula (10), may be written

$$(i\,|\,g\,|\,f)^{(0)} = \frac{1}{A(A-1)} \cdot \frac{1}{16}\,[(i\,|\,1\,|\,f)\,\varrho(\vec{x}^{(1)})\,\varrho(\vec{x}^{(2)}) - (i\,|\,P_\sigma P_\tau\,|\,f)\,|\,\varrho(\vec{x}^{(1)},\vec{x}^{(2)})\,|^2]. \tag{56}$$

Now, one readily finds, with the eigenfunctions (1),

$$\varrho(\vec{x}) = \frac{A}{V}$$
$$\varrho(\vec{x}^{(1)}, \vec{x}^{(2)}) = \frac{A}{V}\,G(r). \tag{57}$$

* *Like* nucleons are nucleons of like charge, irrespective of the spin orientation (*0.2*).

with

$$G(r) = \frac{3}{(\mathrm{r}\varkappa r)^3} (\sin \mathrm{r}\varkappa r - \mathrm{r}\varkappa r \cos \mathrm{r}\varkappa r),$$

and therefore

$$(i\,|g|\,f)^{(0)} = \frac{1}{16\,V^2} \left[(i\,|\,1\,|\,f) - (i\,|P_\sigma P_\tau|\,f)\,G^2(r)\right]. \tag{58}$$

The spatial correlations are, in this approximation,

$$\begin{aligned} g^{(0)}_{=}(r) &= \frac{1}{4\,V^2}\left[1 - G^2(r)\right] \\ g^{(0)}_{\neq}(r) &= \frac{3}{4\,V^2}. \end{aligned} \tag{59}$$

We see that, even in the absence of any nuclear interaction, there is some correlation between congruent nucleons: while non-congruent nucleons are uniformly distributed, congruent ones tend to repel each other (fig. *12.23*): it is this well-known effect of the exclusion principle which, as shown by (58) in connexion with (9), gives rise, in the next approximation, to the exchange energy and thereby to the saturation property of the nuclear forces.

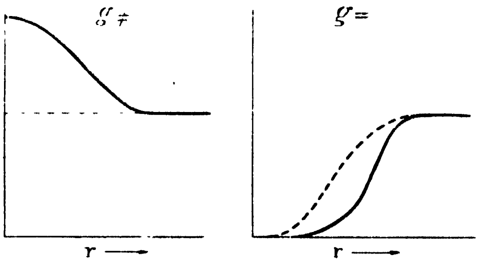

Fig. **12.23.** Spatial correlations between congruent and non-congruent nucleons. The dotted lines represent the initial approximations $g^{(0)}$, in which no account is taken of nuclear interactions.

The next approximation *

$$(i\,|g|\,f)^{(1)} = -\frac{1}{16\,V^2}\cdot\frac{b\,\mathrm{r}}{4\pi^2}\left[(i\,|\,\mathcal{O}\,|\,f)\,G^{(1)}_{\mathrm{ord}}(r) - (i\,|\,\mathcal{O}P_\sigma P_\tau\,|\,f)\,G^{(1)}_{\mathrm{exch}}(r)\right]$$

$$G^{(1)}_{\mathrm{ord}}(r) = \frac{1}{2}\left(\frac{3}{2\pi}\right)^2 \int dv_u\,dv_{u'}\,dv_{u''}\,\frac{\cos \mathrm{r}\varkappa\,\vec{u}'\,\vec{x}}{\vec{u}'\,(\vec{u}' + \vec{u}'' - \vec{u})}\,F(\mathrm{r}\,u') \tag{60}$$

$$G^{(1)}_{\mathrm{exch}}(r) = \frac{1}{2}\left(\frac{3}{2\pi}\right)^2 \int dv_u\,dv_{u'}\,dv_{u''}\,\frac{\cos \mathrm{r}\varkappa\,\vec{u}'\,\vec{x}}{\vec{u}'\,(\vec{u}' + \vec{u}'' - \vec{u})}\cdot F(\mathrm{r}\,|\vec{u}' + \vec{u}'' - \vec{u}\,|)$$

(domain of integration defined by

$$u \leq 1,\ u'' \leq 1,\ |\vec{u} - \vec{u}'| \geq 1,\ |\vec{u}' + \vec{u}''| \geq 1),$$

shows that the nuclear interaction, besides modifying the spatial correlation

* This approximation has been investigated by VOLZ, but only a short account of his results, including fig. *12.23*, has been published (V. WEIZSÄCKER [38], p. 216).

between congruent nucleons, also brings about a spatial correlation between non-congruent nucleons; in fact, using the expression (*8.2–7*) for $\mathcal{O}$, we find for the relevant coefficients

$$\begin{aligned} &\Sigma_{=} (i\,|\,\mathcal{O}\,|\,i) = \Sigma_{=} (i\,|\,\mathcal{O}\,P_\sigma\,P_\tau\,|\,i) = -4\cdot{}^3p = \tfrac{4}{3} \\ &\Sigma_{\neq} (i\,|\,\mathcal{O}\,|\,i) = -3\,({}^3s + {}^1s) - ({}^3p + {}^1p) = -\tfrac{8}{3} \\ &\Sigma_{\neq} (i\,|\,\mathcal{O}P_\sigma\,P_\tau\,|\,i) = +3\,({}^3s + {}^1s) - ({}^3p + {}^1p) = 6\,(1+q) - \tfrac{8}{3}, \end{aligned} \tag{61}$$

the numerical values following from (*11.33*–12, 13). Accordingly,

$$\begin{aligned} g_{=}^{(1)}(r) &= -\frac{1}{4\,V^2}\cdot\frac{b\,r_0}{4\,\pi^2}\cdot\tfrac{1}{3}\,(G^{(1)}_{\text{ord}} - G^{(1)}_{\text{exch}}) \\ g_{\neq}^{(1)}(r) &= +\frac{1}{4\,V^2}\cdot\frac{b\,r_0}{4\pi^2}\cdot[\tfrac{2}{3}\,(G^{(1)}_{\text{ord}} - G^{(1)}_{\text{exch}}) + \tfrac{3}{2}\,(1+q)\,G^{(1)}_{\text{exch}}]. \end{aligned} \tag{62}$$

Clearly, $G^{(1)}_{\text{ord}} > G^{(1)}_{\text{exch}} > 0$; since, moreover, $b > 0$ and $q > 0$, we see that the effect of nuclear interaction is to widen the "hole" in the spatial distribution of congruent nucleons around a given one, while non-congruent nucleons tend to accumulate in the vicinity of this nucleon (fig. *12.23*). In other words, the density fluctuations resulting from the nuclear interactions favour the formation of *clusters*, the simplest (and therefore most frequent) of which consist of four nucleons in the four different spin and isotopic states; such clusters, which have a structure akin to that of α-particles, we shall call *α-clusters*.

The second order contribution $\mathcal{E}^{(2)}$ to the nuclear energy is intimately connected with the spatial correlations between the constituent nucleons. By inserting (60) into (9) and expressing the $G^{(1)}$'s in terms of the $g^{(1)}$'s with the help of (62), we can indeed put $\mathcal{E}^{(2)}$ into the form

$$\mathcal{E}^{(2)} = \tfrac{1}{2}\cdot\frac{A\,(A-1)}{2}\int J(r)\,dv^{(1)}\,dv^{(2)}\,[a_{=}\,g_{=}^{(1)}(r) + a_{\neq}\,g_{\neq}^{(1)}(r)]; \tag{63}$$

the coefficients $a_{=}$, $a_{\neq}$ are easily calculated, taking account of (21):

$$\begin{aligned} a_{=} &= \frac{1+3q+7q^2+9q^3}{1+q} \\ a_{\neq} &= -\frac{1+q^2}{1+q}. \end{aligned} \tag{64}$$

The negative correlation function $g_{=}^{(1)}$ has a positive coefficient, and vice versa. The fact that $\mathcal{E}^{(2)}$ is thus always negative receives a simple interpretation in that the formation of clusters has obviously the effect of increasing the binding energy (in absolute value). As a matter of fact, one may say that the reason why the above estimate of $\mathcal{E}^{(2)}$ leads to such a poor quantitative result is that the individual nuclear model used as a starting point, although implying a tendency to α-clustering, does not

allow one (so to speak) to take advantage of this feature in calculating the binding energy: the strong binding of the particles forming an α-cluster is not properly taken into account. In fact, it is just this clustering which allows the full saturation of the nuclear bonds to come into play and thereby accounts for by far the greatest part of the binding energy. In the nucleus ${}^{16}_{8}O$, for instance, from a total mass-defect of 127,16 MeV an amount of 112,80 MeV could be attributed to four α-clusters with the same binding as normal α-particles. The question therefore naturally arises whether a model in which the existence of α-clusters would be recognized from the outset would not yield better results. This leads us to an examination of the α-particle model and other models of the collective type (*9.22*), which will be dealt with in the next Chapter.

On account of its connexion with the density fluctuations or spatial correlations of the nuclear system, the second order energy $\mathcal{E}^{(2)}$ has been termed by Euler *fluctuation energy* and by Wigner *correlation energy*.

The correlation energy has especially been discussed by WIGNER [34*a*, *b*] in connexion with the problem, formally very similar, of the metallic state. In this case, one has simply (for $\mathcal{N}$ electrons)

$$\mathcal{E}=\mathcal{K}+\frac{\mathcal{N}(\mathcal{N}-1)}{2}\int\frac{e^2}{r}\,dv^{(1)}dv^{(2)}\{[g_{=}^{(0)}(r)+g_{\neq}^{(0)}(r)]+\tfrac{1}{2}[g_{=}^{(1)}(r)+g_{\neq}^{(1)}(r)]\}$$

and the correlation turns out to be negative for electrons with opposite spins as well as for those of like spins, i.e. any two electrons tend to keep as far apart from each other as possible. As a result, the electrostatic repulsion between them is decreased, and the correlation energy accordingly means a tightening of the binding. The effect of correlation is also of some importance, as pointed out by GOMBÁS [43], in calculations of the energy of the stationary states of heavy atoms and ions by means of the Thomas–Fermi method: the correlation energy here appears as a generalization of the polarization effect met with in the treatment of the Helium atom (BETHE [33], p. 339).

12.24. *The virial theorem.* As in atomic theory, the virial theorem yields an exact relation between the mean kinetic energy $\mathcal{K}$ and the potential energy of a nuclear system; its usefulness in this field has first been emphasized by HILL [37]. For a potential of the type

$$\mathcal{V}_{\text{nucl}}=\tfrac{1}{2}\sum_{i,k}\mathcal{V}^{(ik)}(\varkappa r^{(ik)}),$$

the virial theorem can be expressed in the form (WATANABE [39])

$$\mathcal{K}=\tfrac{1}{2}\,\text{av}\sum_{l}(\vec{x}^{(l)}\,\text{grad}^{(l)})\,(\tfrac{1}{2}\sum_{i,k}\mathcal{V}^{(ik)})=\tfrac{1}{2}\,\text{av}\,\varkappa\frac{d}{d\varkappa}\,\mathcal{V}_{\text{nucl}}. \tag{65}$$

Now, we may certainly assume that the distance dependence of the attractive nuclear potential will be such as to imply, quite generally,

$$\text{av}\,\frac{d}{d\varkappa}\,(\varkappa\,|\,\mathcal{V}_{\text{nucl}}\,|)<0\,; \tag{66}$$

therefore

$$\mathcal{K}>\tfrac{1}{2}\,|\,\mathcal{V}\,|. \tag{67}$$

This inequality seems indeed to hold generally for actual nuclei. Since the mean kinetic energy per nucleon K, in the initial approximation of the Fermi gas model, is about equal to the empirical value of the total energy ε_1 per nucleon, the ratio $\mathcal{K}/|\overline{\mathcal{V}}|$ should be about $\frac{1}{2}$ in this approximation in order to yield the correct binding energy; and any improvement on this approximation will tend to increase it (*9.41*).

12.3. Penetration of fast nucleons into heavy nuclei

12.30. With the view of interpreting the disintegration "stars" observed in photographic emulsions exposed to cosmic radiation (*7.14*), HEISENBERG [37] has performed a detailed analysis of the collision between a heavy (Ag or Br) nucleus and a very fast nucleon (with an energy of a few 100 MeV, say). In passing through the nucleus, the impinging particle will communicate part of its energy to several constituent nucleons, which may be knocked out of the nucleus and thus produce in the photographic emulsion a "star" of tracks diverging from the spot occupied by the nucleus. Owing to the strong coupling between the nuclear particles, only those nucleons will actually be able to leave the nucleus which have acquired a sufficient energy in a collision taking place near the surface. The energy distribution of these nucleons, which can be derived empirically from a detailed study of the tracks (WAMBACHER [40]), will primarily depend on the cross-sections for the collisions between the impinging fast nucleon and any one of the constituent nucleons, the states of motion of which may be assumed to form a Fermi distribution at the absolute zero of temperature. Our first task will therefore be to calculate this cross-section for a given transfer of momentum.

12.31. *Differential collision cross-section.* In computing the differential cross-section, we shall use the non-relativistic expression (*8.2*–1) for the nuclear potential; this is permissible as long as the momentum transfer is small compared with M, — a condition not very well fulfilled for the largest momentum transfers possible with an impinging nucleon of a few 100 MeV energy. For such large momentum transfers it would be desirable to carry out a more refined calculation with the help of the meson field theory of the nuclear interaction; however, we shall presently restrict ourselves to cases in which large momentum transfers occur with negligible probability. We shall further assume that we have to deal with a standard heavy nucleus, and we shall neglect any electrostatic interaction with the impinging nucleon. Since this nucleon is a very fast one, Born's approximation may be used, and the differential cross-section takes the form

$$d\mathcal{S} = \frac{c}{v_i^{(1)}} \cdot V \cdot \frac{2\pi}{\hbar} |(i|\mathcal{V}_{\text{nucl}}|f)|^2 \, dn_f; \tag{1}$$

V is the volume of the nucleus, $v_i^{(1)}$ the initial velocity of the impinging particle, dn_f the number of stationary states per unit energy of the system

after the collision, and $(i|V_{\text{nucl}}|f)$ the matrix-element of the nuclear potential, which, according to (*12.11*–4), can be written

$$(i\,|\,V_{\text{nucl}}\,|\,f) = J\,w_p(p)\,(i\,|\,O\,|\,f) - J\,w_p(p')\,(i\,|\,O\,P_\sigma\,P_\tau\,|\,f) \tag{2}$$

with
$$\vec{p} = \vec{p}_i^{(1)} - \vec{p}_f^{(1)} \quad , \quad \vec{p}' = \vec{p}_i^{(1)} - \vec{p}_f^{(2)}, \tag{3}$$

the momenta further satisfying the conservation law

$$\vec{p}_i^{(1)} + \vec{p}_i^{(2)} = \vec{p}_f^{(1)} + \vec{p}_f^{(2)}. \tag{4}$$

We are at liberty to take either of the quantities $\vec{p}$ or $\vec{p}'$ as representing the transfer of momentum involved in the particular collision we are considering; the initial momenta being given, either $\vec{p}$ or $\vec{p}'$ (together with the spin and charge variables) completely specifies the final state. In fact, by (3) and (4), we have, e.g.,

$$\vec{p}_f^{(1)} = \vec{p}_i^{(1)} - \vec{p} \quad , \quad \vec{p}_f^{(2)} = \vec{p}_i^{(2)} + \vec{p}. \tag{5}$$

Let us introduce polar coordinates in momentum space, choosing $\vec{p}_i^{(1)}$ as polar axis and measuring the azimuths from the plane $(\vec{p}_i^{(1)}, \vec{p}_i^{(2)})$. The angular coordinates of the vectors $\vec{p}_i^{(2)}, \vec{p}$ may be denoted by $(\Theta, 0)$ and (ϑ, φ), respectively; we put

$$\zeta = \cos\vartheta, \qquad z = \cos\Theta \tag{6}$$

and get from (5)

$$\begin{aligned} p_f^{(1)2} &= p_i^{(1)2} + p^2 - 2\,p\,p_i^{(1)}\,\zeta \\ p_f^{(2)2} &= p_i^{(2)2} + p^2 + 2\,p\,p_i^{(2)}\,[z\zeta + \sqrt{1-z^2}\,\sqrt{1-\zeta^2}\,\cos\varphi]. \end{aligned} \tag{7}$$

On account of (7), the equation of energy conservation *

$$E_i^{(1)} + E_i^{(2)} = E_f^{(1)} + E_f^{(2)} \tag{8}$$

establishes a relation between p, φ, ζ and the initial momenta; consequently, p' can in principle be expressed in terms of the initial momenta and of two of the variables p, φ, ζ. In particular, for $\vec{p}_i^{(2)} = 0$, the right-hand side of (8) depends only on p and ζ, so that, for a given momentum transfer p, the number dn_f of final states per unit energy is simply

$$\begin{aligned} dn_f &= \frac{V}{(2\pi\hbar)^3}\,p^2\,dp\,d\varphi\,\left|\frac{d\zeta}{d(E_f^{(1)} + E_f^{(2)})}\right| \\ &= \frac{V}{(2\pi\hbar)^3}\,p^2\,dp\,d\varphi \cdot \frac{E_f^{(1)}}{p_i^{(1)}} \\ &= \frac{V}{(2\pi\hbar)^3}\,p\,dp\,d\varphi \cdot \frac{c}{v_i^{(1)}} \cdot \left(1 - \frac{\sqrt{M^2+p^2} - M}{E_i^{(1)}}\right). \end{aligned} \tag{9}$$

* It will be convenient here to use relativistic energies, denoted by the letter E.

on account of (8) and (7); for small momentum transfers, $p \ll p_i^{(1)}$, the factor between brackets reduces to unity. The value of ζ in terms of p and $p_i^{(1)}$, derived from (8), shows that in the case of small p,

$$\zeta \ll 1 \qquad (\text{for } p \ll p_i^{(1)}); \tag{10}$$

this means that in most collisions (for small momentum transfers are the most frequent), while the deviation of the incident fast particle is only very small, a struck nucleon initially at rest recoils in a direction almost at right angles to the path of the fast particle. Since in our case all $p_i^{(2)} \ll p_i^{(1)}$, this conclusion (10) remains valid when the restriction $\vec{p}_i^{(2)} = 0$ is dropped; likewise, the expression (9) for dn_f remains approximately true.

The absolute square of the matrix-element (2) contains the absolute squares of its two terms, with coefficients $|(i|\mathcal{O}|f)|^2$ and $|(i|\mathcal{O}P_\sigma P_\tau|f)|^2$, and two cross-terms, with coefficients $(i|\mathcal{O}|f)(f|\mathcal{O}P_\sigma P_\tau|i)$ and $(i|\mathcal{O}P_\sigma P_\tau|f)(f|\mathcal{O}|i)$. If we are only interested in the average cross-section for any spin or charge state of the interacting nucleons, we have to sum over the different final spin and charge states and to average over the initial states. This amounts to replacing the coefficients of the absolute squares just mentioned by the trace $\frac{1}{16}\,\mathrm{tr}\,\mathcal{O}^2$ and those of the cross-terms by $\frac{1}{16}\,\mathrm{tr}\,\mathcal{O}^2 P_\sigma P_\tau$. A considerable simplification is obtained by adopting Watanabe's device (*12.21–23*) of putting the latter quantity equal to zero, and thus getting rid of the cross-terms. The value of $\frac{1}{16}\,\mathrm{tr}\,\mathcal{O}^2$ is then given by (*12.21–25*). Using the notation (*12.11–3*), we may therefore write the average differential cross-section in the general form

$$d\mathcal{S} = 0{,}52\,\mathfrak{D}^2 \left\{ \left[\frac{F(p/\hbar\varkappa)}{\hbar\varkappa}\right]^2 + \left[\frac{F(p'/\hbar\varkappa)}{\hbar\varkappa}\right]^2 \right\} \cdot \left(1 - \frac{\sqrt{M^2+p^2}-M}{E_i^{(1)}}\right) p\,dp\,d\varphi, \quad (1$$

with

$$\mathfrak{D} = \frac{c}{v_i^{(1)}} \cdot \frac{J}{\hbar\varkappa} \cdot \frac{1}{\varkappa}. \tag{12}$$

For small momentum transfers, i.e. $p \ll p_i^{(1)}$, the contribution from the second term, involving p', will be much smaller than that from the first term, since p' will be larger than p and the Fourier amplitude $F(p/\hbar\varkappa)$ is a decreasing function of its argument. Formula (11) therefore takes in this case the simpler form

$$d\mathcal{S} \approx 0{,}52\,\mathfrak{D}^2 \left[\frac{F(p/\hbar\varkappa)}{\hbar\varkappa}\right]^2 p\,dp\,d\varphi \qquad (p \ll p_i^{(1)}). \tag{13}$$

For a collision of small momentum transfer involving a definite spin and charge transition, the differential cross-section is obtained simply by replacing in (13) the average factor 0,52 by $|(i|\mathcal{O}|f)|^2$. The collision process can then be analyzed somewhat more closely. If account is taken of the saturation conditions (*11.33–7*), the factor $|(i|\mathcal{O}|f)|^2$ can be split into a function of the spins only and a further factor $|(i|\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}|f)|^2$

depending only on the charges of the colliding nucleons. If we average this last factor over the initial isotopic states of the struck nucleon, we find $\frac{1}{2}\sum_{j}|(i|\tau_j^{(1)}|f)|^2$ independently of the final isotopic state of the nucleon. This means that after the collision, the struck nucleon is as likely to be in a proton as in a neutron state. On the other hand, the relative probabilities of the impinging nucleon remaining or not in the same isotopic state after the collision are as 1 to 2.

12.32. *Total collision cross-section.* Let us now calculate the total cross-section $\bar{\mathcal{S}}(\mathsf{E})$ for all collisions in which the final kinetic energy of the *slower* nucleon is $\geq \mathsf{E}$. The condition $p_f^{(2)} \geq P$, P being the momentum corresponding to the energy E, fixes the lower limit p_0 of integration over p; if P is assumed to be small compared with $p_i^{(1)}$, we may use (10) and get from (7)

$$p_0^2 = P^2 - p_i^{(2)2} - 2\,p_0\,p_i^{(2)}\sqrt{1-z^2}\cos\varphi \tag{14}$$

i.e.

$$p_0 \approx -\,p_i^{(2)}\sqrt{1-z^2}\cos\varphi + \sqrt{P^2 - p_i^{(2)2}(\sin^2\varphi + z^2\cos^2\varphi)}\,. \tag{15}$$

The upper limit of integration over p is the absolute value of the momentum transfer $\vec{p}_{\frac{1}{2}}$ corresponding to an equal distribution of momentum in the final state, $\vec{p}_f^{(1)} = \vec{p}_f^{(2)}$. We have thus to integrate $d\bar{\mathcal{S}}$ over p from p_0 to $p_{\frac{1}{2}}$, and average the result over the Fermi distribution of the initial momentum $p_i^{(2)}$.

The calculation can most easily be performed upon the assumption that large momentum transfers play only a negligible role; for this means that one may use the reduced form (13) for $d\bar{\mathcal{S}}$ and integrate over p up to infinity instead of $p_{\frac{1}{2}}$, which affords considerable simplification. On the other hand, this assumption is only fulfilled for types of nuclear potentials involving a sufficiently fast decrease with increasing distance, i.e. a sufficiently fast decrease of the Fourier coefficients $F(p/\hbar\varkappa)$ with increasing p. We shall presently see that this condition means a considerable restriction in the application of the theory, and it would certainly be desirable to extend the calculations beyond their present scope. Starting from formula (13), we write

$$\begin{aligned} \int_{p_0}^{\infty} d\bar{\mathcal{S}} &\quad 0{,}52\,\mathfrak{D}^2\,\mathcal{G}(p_0/\hbar\varkappa)\,d\varphi\,, \\ \mathcal{G}(x) &= \int_x^{\infty} F^2(u)\,u\,du\,; \end{aligned} \tag{16}$$

accordingly,

$$\bar{\mathcal{S}}(\mathsf{E}) = 0{,}52\,\mathfrak{D}^2\cdot\tfrac{1}{4}\int_{-1}^{+1}dz\cdot\frac{3}{p_m^{(0)3}}\int_0^{p_m^{(0)}} p_i^{(2)2}\,dp_i^{(2)}\int_0^{2\pi}\mathcal{G}(p_0/\hbar\varkappa)\,d\varphi\,, \tag{17}$$

p_0 being given by (15), and $p_m^{(0)}$ denoting the maximum momentum of the

zero-point distribution, as given by (*12.13*–10, 8); the corresponding maximum kinetic energy will be called $E_m^{(0)}$. Expressing momenta in terms of $p_m^{(0)}$, we put

$$y \equiv \frac{p_i^{(2)}}{p_m^{(0)}} \qquad \text{and} \qquad \eta \equiv \frac{p}{p_m^{(0)}} = \sqrt{\frac{E}{E_m^{(0)}}}. \tag{18}$$

Formula (17) then takes the form

$$\mathcal{S}(E) = 0{,}52 \cdot \tfrac{3}{2}\, \mathcal{D}^2 \int_{-1}^{+1} dz \int_0^1 y^2\, dy \int_0^{2\pi} d\varphi\, \mathcal{G}(r u_0), \tag{19}$$

with $u_0 \equiv p_0/p_m^{(0)}$ given, according to (15) and (18), by

$$u_0 = -y\sqrt{1-z^2}\cos\varphi + \sqrt{\eta^2 - y^2(\sin^2\varphi + z^2\cos^2\varphi)}. \tag{20}$$

This last formula, as well as (19), can be put into a much simpler form by a change of variables indicated by BAGGE [39]: it consists in going over from the polar coordinates (y, Θ, φ) to cylindrical coordinates (λ, ϱ, ψ), the axis of which is the perpendicular to the axis of the polar coordinates in the plane of origin of the azimuths φ: λ is the distance from the origin measured along this axis, ϱ the normal distance from the axis, and ψ the azimuth around it. Therefore,

$$\lambda = y\sin\Theta\cos\varphi, \qquad \varrho = \sqrt{y^2-\lambda^2}, \qquad \cos\psi = \frac{y}{\varrho}\cos\Theta \tag{21}$$

and

$$u_0 = -\lambda + \sqrt{\eta^2-\varrho^2}; \tag{22}$$

further, the integrations in (19) must be replaced by

$$\int_0^{2\pi} d\psi \int_0^1 \varrho\, d\varrho \int_{-\sqrt{1-\varrho^2}}^{+\sqrt{1-\varrho^2}} d\lambda \ldots .$$

Thus, with the new substitutions

$$x = r u_0 = r(-\lambda + \sqrt{\eta^2-\varrho^2})$$
$$u = r\sqrt{1-\varrho^2},$$

(19) is reduced to the form

$$\mathcal{S}(E) = 0{,}52 \cdot \tfrac{3}{2}\, \mathcal{D}^2 f(r,\eta). \tag{23}$$

with

$$\begin{aligned} f(r,\eta) &= 2\pi r^{-3} \int_0^r u\, du \int_{\sqrt{r^2(\eta^2-1)+u^2}-u}^{\sqrt{r^2(\eta^2-1)+u^2}+u} \mathcal{G}(x)\, dx \\ &= \frac{\pi}{4} r^{-3} \int_{r(\eta-1)}^{r(\eta+1)} \left[2r^2(\eta^2+1) - x^2 - \frac{r^4}{x^2}(\eta^2-1)^2\right] \mathcal{G}(x)\, dx, \end{aligned} \tag{24}$$

the second expression for $f(\mathrm{r}, \eta)$ following from the first by elementary transformations.

The cross-section $\overline{\mathcal{S}}(\mathsf{E})$ directly gives the number $\bar{n}(\mathsf{E})$ of secondary particles with a kinetic energy $\geqq \mathsf{E}$ produced on the average within the nucleus by a single impinging nucleon:

$$\bar{n}(\mathsf{E}) = \frac{A}{\pi R^2} \overline{\mathcal{S}}(\mathsf{E}) . \tag{25}$$

However, only a fraction of these secondaries will be able to escape from the nucleus, the rest dissipating their energy in the form of "heat" by collisions with other constituent nucleons. We shall now try to get an idea of the efficiency of such collisions in slowing down a nucleon penetrating into nuclear matter.

12.33. *Energy loss of a fast nucleon in nuclear matter.* The mean energy loss of a fast nucleon of initial velocity $v_i^{(1)}$ per unit of path in nuclear matter is

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} \cdot \frac{1}{2}\int_{-1}^{+1} dz \int_0^{p_m^{(0)}} \frac{3\, p_i^{(2)2}\, dp_i^{(2)}}{p_m^{(0)3}} \int (E_f^{(2)} - E_i^{(2)})\, d\mathcal{S} , \tag{26}$$

$d\mathcal{S}$ being given by (11), and the integration over p extending from a lower limit defined (on account of the exclusion principle) by $p_f^{(2)} = p_m^{(0)}$ up to the upper limit $p_{\ddagger}$. With the same restriction as above, we may replace $d\mathcal{S}$ by its reduced expression (13) and the upper limit of momentum transfer by infinity; using the unrelativistic expressions for the energies $E_f^{(2)}$ and $E_i^{(2)}$, and taking account of (10), we can then put (26) into the form

$$\begin{aligned} \frac{dE_i^{(1)}}{ds} = &-\frac{A}{V}\, \mathsf{E}_m^{(0)} \cdot \frac{3}{2} \cdot 0{,}52\, \mathcal{D}^2 \int_{-1}^{+1} dz \int_0^1 y^2\, dy \int_0^{2\pi} d\varphi \; \cdot \\ &\cdot \int_{y^2+u^2+2uy\sqrt{1-z^2}\cos\varphi}^{\infty} F^2(\mathrm{r}\,u)\, \mathrm{r}^2\, u\, du\, [u^2 + 2\,u\,y\sqrt{1-z^2}\cos\varphi] ; \end{aligned} \tag{27}$$

going over to Bagge's variables (21), we get

$$\begin{aligned} \frac{dE_i^{(1)}}{ds} = &-\frac{A}{V}\, \mathsf{E}_m^{(0)} \cdot \frac{3}{2} \cdot 0{,}52\, \mathcal{D}^2 \cdot \frac{2\pi}{\mathrm{r}^2} \int_0^1 \varrho\, d\varrho \int_{-\sqrt{1-\varrho^2}}^{+\sqrt{1-\varrho^2}} d\lambda \; \cdot \\ &\cdot \int_{\mathrm{r}(-\lambda+\sqrt{1-\varrho^2})}^{\infty} F^2(u)\, u\, du\, (u^2 + 2\,\mathrm{r}\,\lambda\, u) . \end{aligned} \tag{28}$$

After performing the variable substitution

$$x' = \mathrm{r}\,(-\lambda + \sqrt{1-\varrho^2}), \qquad u' = \mathrm{r}\sqrt{1-\varrho^2}$$

and some obvious partial integrations, and using (*12.13*–10), this finally becomes

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} \cdot 0{,}52\, \mathcal{D}^2 \cdot \frac{2\pi\, (\hbar\varkappa)^2}{M} \cdot \mathcal{H} \tag{29}$$

with

$$\mathcal{H} = \tfrac{1}{2} \int_0^\infty F^2(u)\, u^3\, du\,. \tag{30}$$

It is remarkable that we would have obtained exactly the same result if we had assumed all struck nucleons to be initially at rest. In fact, we should then write

$$\frac{dE_i^{(1)}}{ds} = -\frac{A}{V} \cdot \int_{E_f^{(2)}=M}^{E_f^{(2)}=\infty} (E_f^{(2)} - M)\, d\bar{S}$$

$$= -\frac{A}{V} \cdot 0{,}52\, \mathfrak{D}^2 \cdot 2\pi \int_0^\infty \frac{p^2}{2M} \left[\frac{F(p/\hbar\varkappa)}{\hbar\varkappa}\right]^2 p\, dp\,,$$

which is just (29), (30) *. This result is in conformity with a general property first enunciated by WILLIAMS [38]: *the mean energy loss per unit path of a fast nucleon within a nucleus is the same as though the constituent nucleons were at rest and independent of each other, irrespective of their actual interactions and states of motion.* The physical reason for this behaviour is primarily the short range of nuclear interaction. When considering the passage of a fast nucleon of velocity βc at a distance r from a constituent nucleon, one has to compare the "time of collision" $(r/\beta c)\sqrt{1-\beta^2}$ with the "natural period" $2r_0/\bar{v}$ of the bound nucleon; the latter time is that necessary for the nucleon of mean velocity v to cross the "cell" of dimensions $\approx 2r_0$ within which it is on the average confined by its binding to the other nucleons. If the time of collision is long compared to the natural period, we have "adiabatic" conditions and the course of the collision is essentially affected by the state of binding of the constituent nucleon; in the opposite case, when we have a "sudden impulse", the collision takes place as if the colliding particles were free. This last behaviour thus prevails as long as the distance of closest approach r is smaller than some critical distance of the order of magnitude $\dfrac{\beta}{\sqrt{1-\beta^2}} \cdot \dfrac{c}{v} \cdot 2r_0$. But if the impinging nucleon is very fast ($\beta \approx 1$), this critical distance is many times larger than the range of the interaction between the colliding particles. As the distance of approach increases, any interaction will therefore altogether cease long before adiabatic conditions could at all set in: inside the range of interaction, we have just the conditions of sudden impulse. This argument makes it clear that Williams' theorem should be of very wide scope indeed.

It will be observed that the integral $\mathcal{H}$ exists only if $F(u)$ decreases faster than u^{-2} for large u; this limitation on the admissible types of nuclear potentials is a necessary condition for the validity of our simplified

* BAGGE [39] verifies the same property for the special case of a Gauss potential in an unnecessarily complicated manner.

treatment (ROSENFELD [45a]). In fact the breakdown of the theory caused by the divergence of $\mathcal{H}$ cannot, in view of Williams' theorem, be attributed to the use of the Fermi gas model; nor is there any cogent reason to doubt the legitimacy of Born's approximation in calculating the collision cross-section. The divergence of $\mathcal{H}$ is an indication that we have to take a more careful account of the contributions from collisions with large momentum transfers. The table *12.33*, which contains the explicit expressions for the quantities $F(u)$, $\mathcal{G}(x)$ and $\mathcal{H}$ for various types of nuclear potentials, shows that the restriction imposed upon the choice of the distance dependence of the potential by the convergence of $\mathcal{H}$ is a very severe one; in particular, it is unfortunate that the meson potential has to be excluded.

In the case of the well and meson potentials, the logarithmic divergence of $\mathcal{H}$ can be traced to the occurrence of an infinite field intensity at some point (at the boundary of the well or at the centre of the meson field of force), as a result of which the frequency of large momentum transfers is increased. To be sure, it would be possible to avoid this by a slight modification of the potential: in order to make $\mathcal{H}$ convergent, we have just to "round off" the shape of the well:

$$\begin{aligned} w(\xi) &= 1 - \tfrac{1}{2}\, e^{\alpha(\xi-1)} && \text{for } 0 \leqq \xi \leqq 1 \\ w(\xi) &= \tfrac{1}{2}\, e^{-\alpha(\xi-1)} && \text{for } 1 \leqq \xi\,, \end{aligned} \tag{31}$$

or to "cut off" the singularity of the meson potential:

$$\begin{aligned} w(\xi) &= \text{const} && \text{for } \xi \leqq \xi_0 \\ w(\xi) &= e^{-\xi}/\xi && \text{for } \xi > \xi_0\,. \end{aligned} \tag{32}$$

But, of course, the convergence of $\mathcal{H}$ alone is not sufficient to justify the application of our approximate treatment. Actually, the Gauss and exponential potentials are about the only important instances of nuclear potentials for which this treatment can be trusted.

12.33. *The quantities $F(u)$, $\mathcal{G}(x)$ and $\mathcal{H}$ for some types of nuclear potentials*

Potential	$w(\xi)$	$F(u)$	$\mathcal{G}(x)$	$\mathcal{H}$
Well	-	$\frac{2}{u^3}[\sin u - u\cos u]$	$\frac{1}{x^4}[\sin^2 x - x\sin 2x + x^2]$	∞
Gauss	$e^{-\xi^2}$	$\frac{\sqrt{\pi}}{2} e^{-u^2/4}$	$\frac{\pi}{4} e^{-x^2/2}$	$\frac{\pi}{4}$
Meson	$e^{-\xi}/\xi$	$\frac{2}{1+u^2}$	$\frac{2}{1+x^2}$	∞
Exponential	$e^{-2\xi}$	$\frac{1}{2}\cdot\frac{1}{(1+\frac{1}{4}u^2)^2}$	$\frac{1}{6}\cdot\frac{1}{(1+\frac{1}{4}x^2)^3}$	$\frac{1}{6}$

12.34. *Range of fast nucleon in nuclear matter.* The right-hand side of equation (29) contains the energy $E_i^{(1)}$ through the factor

$$\left(\frac{c}{v_i^{(1)}}\right)^2 = \frac{E_i^{(1)2}}{E_i^{(1)2} - M^2},$$

which occurs in $\mathfrak{D}^2$ according to (12). The integration of (29) consequently yields for the range $R^{(1)}$ of the nucleon of initial energy $E_i^{(1)}$ in nuclear matter

$$R^{(1)} = \frac{(E_i^{(1)} - M)^2}{M E_i^{(1)}} \cdot \frac{1}{\frac{3}{2} \cdot 0{,}52 \cdot \frac{H}{(\varkappa r_0)^2} \left(\frac{J}{M}\right)^2} \cdot r_0 . \tag{33}$$

Working out this formula for a Gauss potential, with the numerical data (*12.21–29*) and $r_0 = \frac{1}{2}d$, we find

$$R^{(1)} = \frac{(E_i^{(1)} - M)^2}{M E_i^{(1)}} \cdot \frac{1}{2{,}64 \cdot 10^{-3}} \cdot r_0 . \tag{34}$$

The exponential potential (table *6.432*) leads to a very similar result, the factor 2,64 in the denominator being replaced by 2,97. In order to traverse a ^{80}Br nucleus a nucleon must, according to formula (34), have an initial kinetic energy of at least 155 MeV. On the other hand, for a kinetic energy of about 50 MeV, the range amounts to only 0,1 of the diameter of the Br nucleus.

12.35. *Theory of disintegration stars.* In view of the last estimates, we shall assume, following Heisenberg, that only the secondaries released in a superficial spherical shell of thickness $(4\pi/3)^{\frac{1}{3}}\, r_0 \approx 1{,}6\, r_0$ with a momentum directed outward are able to escape from the nucleus, provided their total kinetic energy E is larger than the mean potential energy $|\,\overline{\mathfrak{V}}\,|$ of the nucleus. In the nuclear model here adopted, the latter quantity is $\mathsf{E}_m^{(0)} + \varepsilon$, ε being the absolute value of the mean binding energy of a nucleon (*2.21*); the ejected nucleons of energy $\geqq$ E are therefore observed outside the nucleus to have a kinetic energy $\geqq$ E_0, with

$$\mathsf{E}_0 \approx \mathsf{E} - (\mathsf{E}_m^{(0)} + \varepsilon). \tag{35}$$

Their mean number per impinging nucleon is

$$\begin{aligned} n_0(\mathsf{E}_0) &= \bar{n}(\mathsf{E}) \cdot \tfrac{1}{2} \cdot \frac{\text{volume of shell}}{\text{volume of nucleus}} \\ &= \bar{n}(\mathsf{E}) \cdot \frac{3}{2A^{1/3}} \left[1{,}6 - \frac{(1{,}6)^2}{A^{1/3}}\right] \\ &= \frac{3}{2\pi r_0^2} \cdot \mathcal{S}(E) \left[1{,}6 - \frac{2{,}56}{A^{1/3}}\right]. \end{aligned} \tag{36}$$

owing to (25). Inserting (23) with the expression (12) for $\mathfrak{D}$, we get, for the Gauss potential, and $r_0 = \frac{1}{2}d$,

$$\bar{n}_0(\mathrm{E}_0) = 0{,}09\left(\frac{c}{v_i^{(1)}}\right)^2\left(1{,}6 - \frac{2{,}56}{A^{1/3}}\right) f_G(\mathrm{r}_G,\eta) \qquad (37)$$

$$\mathrm{r}_G \approx 2{,}07\,.$$

The distribution function $f_G(\mathrm{r}_G, \eta)$ can in this case be evaluated explicitly; it was first derived by BAGGE [39]. Putting $\beta_G = \frac{1}{2}\mathrm{r}_G^2$, it may be written as

$$f_G(\mathrm{r}_G,\eta) = g[\sqrt{\beta_G}(\eta-1)] - g[\sqrt{\beta_G}(\eta+1)]$$

with

$$g(x) = \frac{\pi^{5/2}}{64\,\beta_G^{3/2}}\,[4\beta_G^2(\eta^2-1)^2 + 4\beta_G(\eta^2+1) - 1]\,[1-\Phi(x)] \qquad (38)$$

$$- \frac{\pi^2}{32\,\beta_G^{3/2}}\left[1 + \frac{2\beta_G^2}{x^2}(\eta^2-1)^2\right] x e^{-x^2}$$

and

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\,.$$

For $\eta \gtrsim 1$, $\beta_G > 2$, it reduces approximately to $g[\sqrt{\beta_G}(\eta - 1)]$, while for $\eta \gg 1$, one gets, with the help of the asymptotic expansion of $\Phi(x)$,

$$f_G(\mathrm{r}_G,\eta) \simeq \frac{\pi^2\eta}{8\,\beta_G^2(\eta-1)^3}\, e^{-\beta_G(\eta-1)^2}. \qquad (39)$$

Take, e.g., a nucleon of initial kinetic energy ≈ 465 MeV (i. e. $E_i^{(1)}/M = 3/2$) colliding with a ${}^{80}_{35}\mathrm{Br}$ nucleus; the minimum kinetic energy of the expelled secondaries is, according to (35), $\mathrm{E}_m + \varepsilon$ with $\varepsilon \approx 8{,}6$ MeV, corresponding to $\eta_{\min} = \sqrt{1 + \frac{\varepsilon}{\mathrm{E}_m}} = 1{,}16$ (*9.41*–5). Since, moreover,

$$\left(\frac{c}{v_i^{(1)}}\right)^2 = 1{,}8\,, \qquad 1{,}6 - \frac{2{,}56}{A^{1/3}} = 1{,}01\,, \qquad f_G(2{,}07\,;\,1{,}16) \approx 0{,}88\,,$$

we get from (37) for the total average number of secondaries per collision

$$\bar{n}_0(0) \approx 0{,}14\,; \qquad (40)$$

this represents $0{,}35 \cdot f_G(2{,}07;\,1{,}16)/f_G(2{,}07;\,1) = 0{,}35 \cdot 0{,}88/1{,}44 = 0{,}22$ of the total number of secondaries generated within the nucleus.

The experimental material has been carefully analyzed (BAGGE [39], ORTNER [40]) with a view to obtaining statistics of the numbers of particles of definite energy composing the disintegration stars. The order of magnitude (40) derived for the total number of particles seems rather too low to account for the results of observation. This discrepancy led BAGGE [39] to suggest a somewhat less direct mechanism of production

of the stars: the impinging cosmic ray particle would, by an "explosive" process, give rise to several nucleons of high energy, which would then release by intranuclear collisions the particles constituting the star. This hypothesis would in any case uphold the *proportionality* between the mean number of star particles of energy $\gtrsim E_0$ and the function $f(r, \eta)$, which characterizes the underlying collision process and in particular embodies the dependence of its cross-section on the *range* of the nuclear potential. Such a proportionality

$$n_0(E_0) \sim f(r, \eta) \tag{41}$$

is all that is required to derive from the observed numbers of particles $\bar{n}_0(E_0)$ a determination of the range.

12.36. *Determination of range of nuclear force.* The available statistics (ORTNER [40]) extend from $E_0 = 10$ MeV to $E_0 = 50$ MeV; taking again $\varepsilon = 8{,}6$ MeV and allowing for the variation of E^0_m around E_m for varying r_0, we are interested in values of η lying in the interval $1{,}1 \lesssim \eta \lesssim 1{,}85$, while we expect r_G to lie between 2 and 2,2. Now, when η and r_G vary within these intervals, it appears that the function $f_G(r_G, \eta)$ can to a fair approximation be represented by

$$f_G \sim 10^{-(0{,}233 + 0{,}15\,\beta_G)\,\eta^2}. \tag{42}$$

By (*12.13*–10), (18), (35), we have

$$\beta_G = \tfrac{1}{2}\, r_G^2 = \frac{M}{\hbar^2 \varkappa^2}\, E_m^{(0)}, \qquad \eta^2 = \frac{E_0}{E_m^{(0)}} + \text{const},$$

so that *

$$\log_{10} f_G = \text{const} - 0{,}233\, \frac{E_0}{E_m} \left(\frac{2\, r_0}{d}\right)^2 - 0{,}15\, \frac{1}{(\varkappa d)^2} \cdot \frac{E_0}{\hbar^2/d^2 M},$$

or, expressing E_0 in MeV,

$$\log_{10} f_G = \text{const} - E_0 \left[0{,}0288\, \frac{1}{(\varkappa d)^2} + 0{,}0096 \left(\frac{2\, r_0}{d}\right)^2 \right]. \tag{43}$$

It will be observed that, although f_G strictly speaking involves not only $\varkappa$ but also r_0, it is so insensitive to variations of the latter parameter as to provide a practically unique determination of the former; at any rate, any uncertainty as to the proper choice of the numerical value of r_0 is by far outweighed by the margin of error of the experimental data **.

The range determination for the Gauss potential with the help of (43) has already been discussed in *7.14*.

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* It seems as though the difference (35) between E_0 and E had disappeared from the expression for $\log_{10} f_G$; but actually, the numerical coefficients would be different for another range of variation of η.

** This point is not sufficiently stressed by Heisenberg nor by Bagge.

CHAPTER XIII

COLLECTIVE NUCLEAR MODELS

13.1. The α-particle model

13.10. The occurrence of α-clusters (*12.23*) in a nucleus is, of course, something quite different from a structure built up of α-particles keeping their individuality: the α-clusters lack the permanence of such α-particles: the nucleons which at some time have joined to form an α-cluster will soon separate and enter other transient clusters. Even so, one might presume that the life-time of an α-cluster would be sufficiently long to cause the nuclear structure to bring out at least some features of a model consisting of an assembly of permanent α-particles (and additional loose nucleons) bound together by suitably defined forces: compared with the individual models, such an idealisation would approximate the true nuclear constitution so to speak from an opposite direction. In the early work along these lines, the interaction between the α-particles was assumed to have the properties of the van der Waals attraction between the molecules of a liquid; in particular, most of the argument was based on the assumption that the α-particle forces were additive, i.e. that the total interaction of a system of α-particles was the sum of the interactions of adjacent pairs. A closer investigation, however, shows that this will not be the case unless the elementary law of interaction between two nucleons involves a sufficiently strong residual attraction of the "ordinary" (non-exchange) type and of longer range than the bulk of the force. If this rather artificial assumption is not made, it is found that the van der Waals force between two α-particles, in sharp contrast with that between molecules, has a *shorter* range than the first order "exchange" forces; and this fundamental difference makes the analogy of the assembly of α-particles with a liquid droplet quite untenable. The resulting situation would rather suggest a comparison of α-particle interactions with the chemical binding forces between atoms: just as the latter arise from exchange of electrons between atomic systems, the former are primarily conditioned by the exchange of nucleons between the α-clusters. The properties of a nucleus envisaged as a quasi-molecular aggregate of α-particles are, then, not so rigidly dependent on the conception of these particles as permanent units: its state of motion, e.g., will be describable in terms of rotations and vibrations, and for a legitimate use of such a description, it will suffice that the periods of these motions be short in comparison with the life-time of an α-cluster: this condition will allow us to determine the validity of the conclusions which can be derived from the model. In this section, we shall in turn discuss the different points of the preceding argument.

13.11. *Empirical evidence on α-particle interaction.* We shall begin by an examination of the empirical evidence bearing on the problem of the interaction between two α-particles. This material, which has been subjected to a critical analysis by WHEELER [41*a*, *b*], consists of data on the scattering of α-particles in Helium [41*a*] and on the ^8Be nucleus [41*b*]. On the conception under discussion, the latter nucleus would just consist of two α-clusters and thus be comparable to a symmetric diatomic molecule. However, its ground state is not a state of binding, but a ^1S virtual level of about 0,125 MeV energy. Further, the existence of an excited state of about 2,8 MeV above the ground state, first recognized by Dee and Gilbert, has been well established by subsequent experiments. Additional information on this and other excited states of the system of two α-clusters may be derived from the analysis of the scattering experiments.

The scattering of α-particles by Helium can be treated quite similarly to the proton-proton scattering discussed in *7.11*, with the simplifying circumstance that the identical scattering partners in this case have no intrinsic angular momentum and obey Bose statistics (*1.22*). Accordingly, only the waves of even orbital momentum contribute to the scattering cross-section, and from formula (*7.11*–14), — in which α now means $4e^2c/\hbar v$ owing to the double charge of the α-particle — we readily find that the ratio of observed to purely classical (Rutherford) scattering cross-section can be written as $|\mathfrak{R}|^2$, $\mathfrak{R}$ being the complex quantity

$$\begin{aligned}
\mathfrak{R} &= \mathfrak{R}_c(\Theta) + \mathfrak{R}_c\left(\frac{\pi}{2} - \Theta\right) + \sum_{\text{even } l} \mathfrak{R}_l(\Theta)\left(1 - e^{2i\delta^{(l)}}\right) \\
\mathfrak{R}_c(\Theta) &= \frac{\cos^2\Theta}{\sqrt{\sin^4\Theta + \cos^4\Theta}}\, e^{-i\alpha \log \sin^2\Theta} \qquad (1) \\
\mathfrak{R}_l(\Theta) &= -\frac{2i}{\alpha}\sqrt{\frac{1}{\sin^4\Theta} + \frac{1}{\cos^4\Theta}}\,\sqrt{4\pi(2l+1)}\, Y_l^0(\cos 2\Theta)\, e^{2i\sum_{k=1}^{l} \operatorname{arctg} \alpha/k}\,;
\end{aligned}$$

the two first terms represent pure Coulomb scattering, the interference between them giving rise to the characteristic deviation from Rutherford's formula, first predicted by Mott; the last sum is the contribution from the anomalous scattering due to the proper nuclear interaction between the α-particles *.

In an Argand diagram, $\mathfrak{R}$ is represented by a vector, the length of which is known from observation in terms of scattering angle Θ and energy E of the impinging particles. On the other hand, it appears, according to (1), as the resultant of a number of vectors, of which those representing Coulomb scattering are completely fixed for any given Θ and E, while those representing anomalous scattering are only known in

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* One has $\gamma^{(l)} - \gamma^{(0)} = \sum_{k=1}^{l} \operatorname{arctg} \alpha/k$.

length, their directions depending on the phases $\delta^{(l)}$. Provided we can restrict ourselves to the consideration of the few first values of l (in fact, $l = 0, 2, 4$), it is then possible, by an elegant graphical procedure described by Wheeler, to determine for each energy value one or several sets of possible values of the corresponding phases $\delta^{(0)}$, $\delta^{(2)}$, $\delta^{(4)}$. The analysis of the available experiments, which for a number of scattering angles in the interval 5° ... 45°, extend over a range of energies $\mathsf{E} \approx 1{,}5 \ldots 8{,}5$ MeV, finally leads to two * acceptable sets of phase functions $\delta^{(0)}(\mathsf{E})$, $\delta^{(2)}(\mathsf{E})$, $\delta^{(4)}(\mathsf{E})$.

In order to make a choice between these two, account has to be taken of the excited state of the ^{8}Be-nucleus at about 2,8 MeV: this leads us indeed to expect a resonance phenomenon in the scattering of α-particles of about 5,6 MeV kinetic energy. According to general dispersion theory (for details, cf. WHEELER [41a]), the phase of the partial wave having the same orbital momentum as the state in question will increase by about π in the vicinity of the resonance energy. Only one of the two solutions of the scattering problem satisfies this requirement, and is thus definitively accepted: the phase exhibiting the resonance behaviour is $\delta^{(0)}$, indicating that the Dee-Gilbert level of ^{8}Be, like the ground state, must be a ^1S level; this conclusion, as shown by WHEELER [41*b*], is in agreement with other evidence concerning the yields of nuclear reactions which involve this excited state. Moreover, its half-width can be estimated, by a closer study of the variation of $\delta^{(0)}$ with energy, to be $\Gamma \approx 0{,}8$ MeV (corresponding to a mean life $\hbar/c\Gamma \approx 0{,}8 \cdot 10^{-21}$ sec), also in agreement with previous inferences from nuclear reactions.

The behaviour of $\delta^{(2)}(\mathsf{E})$ points to the existence of a much broader excited ^1D state of ^{8}Be with excitation energy of about 4 to 5 MeV. From the fact that the phase $\delta^{(4)}$ takes appreciable values only for $\mathsf{E} > 6$ MeV, i.e. [by formula (*5.14*–28), applied to the Coulomb potential] for a distance of closest approach shorter than $9 \cdot 10^{-13}$ cm, one can finally infer that this last figure represents an upper limit to the range of interaction between two α-particles.

13.12. *Theory of the interaction between two α-particles.* Unfortunately, there is as yet no complete theoretical treatment of the interaction between two α-particles, and therefore no quite definite conclusion about the character of this interaction can be reached; in fact, on the basis of the fragmentary discussion hitherto performed by various authors, either of two sharply opposed standpoints can still be maintained. The procedure followed in all the published ** investigations of this problem is a simple

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* A third set of phases which also gives a good fit with observation must be rejected because the phases $\delta^{(2)}$, $\delta^{(4)}$ decrease instead of increasing for small increasing values of the energy.

** A treatment based on the method of resonating group structure (*13.3*) has been announced by WHEELER [37*a*], but not published.

application of the Hartree-Fock variational method (*9.21*) to the system of eight nucleons constituting the two α-particles: the vector $\vec{R}$ joining their centres of gravity is, however, — just as in the analogous molecular problem, — regarded as a parameter. The interaction energy between the α-particles is obtained as the difference between the total energy of the system and that of two separate, non-interacting α-particles (calculated by the same method). Let $\vec{x}^{(i)}$ $(i = 1, 2, 3, 4)$ be the radii vectores from the centre of gravity of an α-particle to its constituent nucleons. The Slater-determinant of our ^{8}Be-system, considered as an aggregate of two α-clusters, will then involve only two distinct spatial eigenfunctions of individual nucleons, viz. $\varphi(\vec{x}^{(i)})$ and $\varphi(\vec{R} - \vec{x}^{(i)})$ $(i = 1, 2, 3, 4)$, which must be multiplied by the appropriate spin and isotopic eigenfunctions; since these spatial wave-functions are not orthogonal, the Slater-determinant gets an additional normalizing factor

$$\{1 - [\int \varphi(\vec{x})\, \varphi(\vec{R} - \vec{x})dv]^2\}^{-2}.$$

The result of the first approximation of this calculation will very sensitively depend on the "exchange" properties and distance dependence of the elementary law of interaction between two nucleons. The difference between the *potential* energies of the ^{8}Be system and the two separate α-particles represents an attractive potential for all values of the distance R between the α-particles, with approximately the same range as the nucleon interaction potential; the repulsive contributions from the differences of kinetic and Coulomb energies will clearly predominate both at small distances, owing chiefly to the increase of the kinetic energy of the ^{8}Be system, and at large distances, where only the long-range Coulomb energy subsists. But the behaviour of the interaction energy at intermediate distances will, as emphasized by INGLIS [41], critically depend on the properties of the nucleon interaction potential in the domain of distances exceeding its range. If this potential has a sufficiently strong "tail", or residual intensity, the attractive part of the α-particle interaction may become large enough at such intermediate distances to overcome the Coulomb repulsion; if, moreover, the "tail" of the nucleon potential is of the ordinary type, the resulting net attraction between α-particles will have the additivity property of intermolecular van der Waals forces. In general, however, the outcome will be a first order repulsion at all distances, and in order to account for the binding of α-particles it will be necessary to investigate higher approximations.

As a potential giving rise to additive first order attraction between α-particles, Inglis suggests a superposition of two Gauss potentials of different ranges:

$$V = J\{-\tfrac{1}{4} P_\sigma + \tfrac{3}{4} P_\sigma P_\tau\} e^{-\varkappa^2 r^2} - J' e^{-\varkappa'^2 r^2} \tag{2}$$

with
$$J = 71{,}4\ \text{MeV}, \qquad J' = 3{,}1\ \text{MeV}$$
$$\varkappa^{-1} = 1{,}34 \cdot 10^{-13}\ \text{cm}, \qquad \varkappa'^{-1} = 2{,}24\, \varkappa^{-1}.$$

His only argument for the occurrence of the longer range "tail" is taken from the analysis of proton-proton scattering (*7.13*); in fact, the effective ^1S potential derived from (2) closely approximates the meson potential proposed by Breit to account for the scattering data, i.e. that corresponding to the meson mass $M_m = 326\ m$. Quite apart from the small probability of such a high value of the meson mass, the special form of spin and charge dependence of the operator (2) is widely arbitrary; it is compatible with our first saturation requirement (*11.13, 11.2*), but it is not clear whether it satisfies the second (*11.3*). Inglis only shows that it gives about the right binding energy of the deuteron and the α-particle. It should especially be stressed that there is no reason for assuming the "tail" to be just an ordinary Wigner force. Inglis' argument for first order attraction between α-particles is thus far from convincing.

The straightforward assumption of a nucleon interaction represented by a simple Gauss potential has been discussed by MARGENAU [41]. For the parameters fixing the spin and charge dependence of the nucleon interaction, he takes, in conformity with the saturation requirement (*11.31–2*) and the relation (*11.21–6*) directly connected with the deuteron properties, $a_0 = 0$ and $a_\tau - a_\sigma = 0{,}1$; he further leaves a_σ undetermined and finds that the result of the variational procedure, in the first stage of approximation, is practically independent of this parameter. As trial functions for $\varphi(\vec{x})$ he adopts, as usual, an harmonic oscillator eigenfunction involving a variational parameter. The values of the potential constants minimizing the energy of a single α-particle are

$$J = 35{,}6\ \mathrm{MeV}, \qquad \varkappa^{-1} = 2{,}25 \cdot 10^{-13}\ \mathrm{cm}; \tag{3}$$

as expected (*9.21*), they are not in very good agreement with those derived from the two-nucleon data (*6.432*). The final result, as already stated, is that in the first approximation, the α-particles will show no tendency at all to attract each other: there subsists a repulsion at all distances.

This behaviour finds an explanation of rather general scope (WERGELAND [41]) in the closed structure of the α-particles (analogous to rare gas configurations of atomic systems): mutual attraction can only arise through virtual * "polarisation" or "activation" of such structures, — a feature first appearing in the second approximation **. According to the well-known formalism of perturbation theory, this second approximation may be expressed as $-\mathrm{av}\, V^2/\mathcal{E}_m$, the quantity

* The word "virtual" indicates (*1.321*) that we are not here concerned with actual polarization or activation processes, but only with formal features of the perturbation method involving the matrix-elements of transitions to intermediate states of excitation of the particles, which also essentially determine the probability of actual occurrence of such processes.

** The importance of virtual α-particle activation in the nuclear case, i.e. when the interacting α-particles (in contrast to the case of the He_2 molecule) come close together, has already been stressed in an early paper by ELSASSER [34].

av $\mathcal{V}^2$ being the expectation value of the square of the potential energy of the system, while $\mathcal{E}_m$ denotes a suitable average over its excitation energies. The tedious calculation of av $\mathcal{V}^2$ has not been carried out by Margenau at full length: in the first place, instead of considering the appropriate linear combination of exchange operators $1, P_\sigma, P_\tau, P_\sigma P_\tau$ in the nuclear potential, he treats the four types of potentials separately; moreover, he neglects the exchange terms in av $\mathcal{V}^2$, which means that he only retains the α-particle interactions with the longest range. Within the scope of this estimate, he shows that the second order attractive force between the α-particles has a decidedly *shorter* range than the first order repulsion. This is the essential difference from the case of intermolecular forces: in that case, the analogous term has the longer range and is thus predominant in a certain region of intermediate intermolecular distances, in which it just represents the van der Waals attraction, with the additive property inherent in interactions of the ordinary type. We see that in the α-particle case the situation is completely reversed: exchange forces of any order of approximation will here be expected to play just as essential a part as the ordinary van der Waals attractions, with the result that there can be no additivity of the interactions between more than two α-particles.

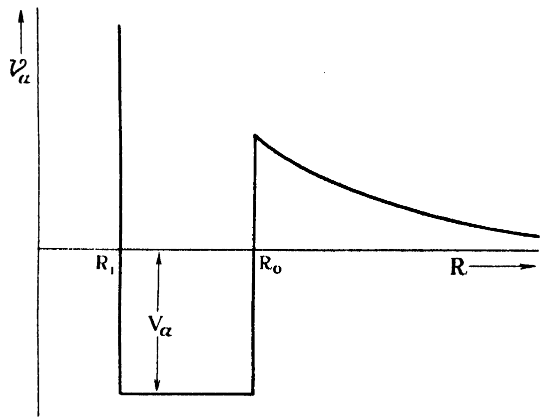

Fig. **13.12.** Idealized potential for a pair of α-particles.

We shall return in a moment to this aspect of the problem. But first we must conclude the discussion of the system of two α-particles, for which the question of additivity does not arise. According to the above results, we shall in any case expect the interaction potential to reduce practically to the Coulomb repulsion at large distances, to become attractive at intermediate distances and to go over very steeply, at still smaller distances, into a strong repulsion. In order to investigate in how far this general behaviour of the interaction is able to account for the observed properties of the system, Margenau adopts the schematic representation given by fig. *13.12*; he states, however, that the results he obtains by means of this rough approximation are not essentially influenced by wide variations in the shape of the potential. For $R > R_0$, a pure Coulomb field is assumed, and the "range" R_0 is fixed at $R_0 = 4{,}5 \cdot 10^{-13}$ cm. The depth V_α is fixed

in terms of R_1 by the condition that there should exist only one stationary state of about zero energy (*13.11*). Then the phase $\delta^{(0)}$ can be calculated for various values of R_1 in the interval $(0, R_0)$: at low energies there is general agreement as to order of magnitude with the result of Wheeler's analysis, but the model fails to yield the increase of $\delta^{(0)}$ in the resonance region. It can be seen that the range R_0 is too small for the type of potential under consideration to give rise to any resonance level of as low energy as ≈ 3 MeV. We here meet with one of the many shortcomings of the α-particle model in respect of quantitative description.

13.13. *Binding energies of α-nuclei.* It is a striking fact, brought out by the work of WEFELMEIER [37*a*, *b*] and of HAFSTAD and TELLER [38], that the binding energies of the α-nuclei would allow of a direct interpretation on the assumption of additive van der Waals interactions between α-particles. The structure of such nuclei, on the α-particle model,

13.13. *Binding energies of α-nuclei*

Nucleus	Number of α-particles	Configuration	Number of bonds	α-binding energy 10^{-4} MU	α-binding per bond 10^{-4} MU
^{8}Be	2	straight line	1	$-0{,}87 \pm 0{,}70$	—
^{12}C	3	triangle	3	$77{,}00 \pm 0{,}87$	$25{,}67 \pm 0{,}29$
^{16}O	4	tetrahedron	6	$154{,}40 \pm 1{,}24$	$25{,}73 \pm 0{,}21$
^{20}Ne	5	trigonal bipyramid	9	$204{,}05 \pm 2{,}16$	$22{,}67 \pm 0{,}24$
^{24}Mg	6	tetragonal bipyramid (octahedron)	12	$310{,}5 \pm 5{,}0$	$25{,}9 \pm 0{,}4$
^{28}Si	7	pentagonal bipyramid	16	365,3 or 387,3 $\pm 5{,}5$	22,8 or 24,2 $\pm 0{,}5$
^{32}S	8	sphenoidal bipyramid	19	$499{,}9 \pm 3{,}2$	$26{,}3 \pm 0{,}2$
^{56}Fe	13	centered icosahedron	42	931 ± 21	22 ± 2

can be compared with that of "geometrical models" consisting of close packings of rigid spheres: each such structure is characterized by a certain number of "bonds", or pairs of adjacent particles, and on the additivity hypothesis the binding energy of the model would simply be proportional to the number of bonds. Empirically, this binding energy can be computed as the difference between the mass of the α-nucleus and the total mass of the constituent α-particles. As the accompanying table shows, almost all the lighter α-nuclei (the only exceptions being ^{8}Be and ^{20}Ne) actually exhibit a surprisingly close proportionality between the empirical "α-binding energies" and the numbers of bonds; also the heavier ^{56}Fe nucleus, which, apart from 4 extra neutrons, would correspond to a symmetrical arrangement of 12 particles around a central one, obeys the same law.

Margenau's results about the interaction between two α-particles rule out any simple explanation of this remarkable regularity. As pointed out by GRÖNBLOM and MARSHAK [39], it must in reality arise from a quite

complicated compensation between energy terms of different orders of approximation, essentially involving non-additive features of α-particle interactions. In the case of the three-α-particle system constituting ^{12}C, for instance, there occur among others in the second approximation, virtual transitions in which a proton from one α-particle and a neutron from another go over into an excited state of the third: and the contribution from this type of transitions turns out to be of the same order of magnitude as that corresponding to second order interactions between pairs of α-particles. The latter will in fact just about compensate the first order repulsion, so that the main effect will be of the non-additive, many-body type; however, since the number of many-particle "bonds" increases more rapidly with the number of α-particles than the number of adjacent pair "bonds", there must also clearly occur partial compensations of the second order contributions by third order terms, and so on. The α-particle model would thus not seem to offer any more promising starting point than the statistical model for a quantitative calculation of nuclear energies.

13.14. *Lability of α-clusters.* Another aspect of the two-α-particle problem, strikingly illustrating the importance of the lability of the α-clusters in the determination of the energy of the system, is brought to light by the method of attack proposed by WERGELAND [41], which avoids the calculation of the second approximation. Observing that the true structure of the ^{8}Be nucleus must be, so to speak, something in between an aggregate of two α-particles and an assembly of eight independent nucleons, Wergeland chooses as trial eigenfunction for the application of the variational principle a linear combination of the eigenfunction Ψ_α of the α-particle model, discussed above (*13.12*), and that, Ψ_s, corresponding to the individual model; in the latter, the individual spatial wave-functions are again harmonic oscillator eigenfunctions, but we have now, besides that of the ground state of the oscillator, which alone occurs in the α-particle model, also to introduce that of the first excited state. We thus put

$$\Psi = \Psi_\alpha + \lambda \Psi_s \tag{4}$$

and consider λ as the variational parameter. The oscillator parameters *, on the other hand, are not varied independently, but related in such a way that for infinite separation of the α-particles ($R \to \infty$), the energy of the α-particle system is about equal to that of the 8-nucleon system: this somewhat arbitrary relation takes account of the fact that in the latter system four of the nucleons, being promoted to an excited oscillator state, have on the average larger momenta than the others. Assuming, for simplicity, a pure Majorana potential of the Gauss type, with the range and (approximately) the strength given by (3), Wergeland finds, under the conditions stated, an extremal value $\lambda = 1{,}34$, together with

$$\Delta = \int \Psi_\alpha \Psi_s = 0{,}304. \tag{5}$$

* Wergeland assumes anisotropic oscillator eigenfunctions, with two parameters.

This result can be interpreted, following a suggestion of v. WEIZSÄCKER [38], by saying that the α-particles are partly "dissociated" in the (liquid) nuclear matter; the "degree of dissociation" on this picture would be given, according to (4), (5), by

$$\delta = 1 - \frac{(\int \Psi_\alpha \Psi)^2}{\int \Psi^2} = \frac{1 - \Delta^2}{1 + 2\lambda\Delta + \lambda^2} \approx 0{,}25. \tag{6}$$

Such a figure once more confirms our previous conclusion as to the unsuitability of the α-particle model for energy computations.

13.15. *Rotations and vibrations of α-nuclei.* Notwithstanding the negative conclusions of the preceding discussion, there is a class of more qualitative predictions for which the model under discussion furnishes a sufficiently firm basis, viz. those which follow from the behaviour of the system under spatial rotations and symmetries. The justification for applying such considerations to an α-particle system depends in fact only on the relative stability of the α-clusters during time intervals larger than the periods of vibration and rotation in the stationary states of the system. Let us consider, for instance, the n-th mode of surface vibration of the nucleus, with wave-length $\lambda \approx 2\pi R/n$ (R denoting the nuclear radius); we assume, of course, that n is small enough for $\lambda/2$ to be larger than the dimensions of an α-cluster ($n \lesssim A^{\frac{1}{3}}$). A classical formula of Lord Rayleigh gives, for the corresponding circular wave-number ω,

$$\omega^2 = n(n-1)(n+2)\,O \cdot \frac{4\pi}{3} \cdot \frac{1}{MA}, \tag{7}$$

whence for the period

$$\tau \approx n^{-\frac{3}{2}} A^{\frac{1}{2}} \cdot 0{,}4 \cdot 10^{-21}\ \text{sec}, \tag{8}$$

if we use the value of the surface tension O deduced from (2.21–5). We may then, as proposed by WHEELER [37*a*], define the lability of an α-cluster in this state of vibration by the time t required by such a cluster to exchange a nucleon with a part of its surroundings moving in opposite phase: this will be, in order of magnitude, the time required for a nucleon to "diffuse" through a distance $\lambda/2$. Classically, we should have a diffusion formula of the type

$$(\lambda/2)^2 = \tfrac{1}{3}\,\Lambda_d\, v\, t,$$

v being the average nucleon velocity and Λ_d a suitably defined mean free path; taking $\bar{v}$ of the order of the limiting velocity v_m of the Fermi distribution and $\Lambda_d \approx r_0$, we should get in this way

$$t \approx n^{-2} A^{\frac{2}{3}} \cdot 0{,}6 \cdot 10^{-21}\ \text{sec}, \tag{9}$$

or

$$t \approx (A^{\frac{1}{3}}/n)^{\frac{1}{2}} \cdot 1{,}5\,\tau\,; \tag{9a}$$

for the modes of vibration we are considering, even this classical estimate would give $t > \tau$. But the potential barriers due to the surrounding nucleons will be very effective in further reducing the rate of diffusion, at least in states of low excitation energy: for the probability of concentration on the diffusing nucleon of the "activation energy" $\approx |\varepsilon| \approx$ 10 MeV (*2.21*) necessary to surmount the barrier will contain a factor $e^{-|\varepsilon|/T}$, the temperature T not exceeding $\approx$ 2 MeV for the excitations obtainable in ordinary transmutation experiments. We may therefore conclude that the condition for treating the nucleus as a system of permanent α-particles is fulfilled when one has to do with vibrational or rotational stationary states of sufficiently low excitation. As the excitation energy increases, however, the α-clusters will tend gradually to dissolve and the nuclear properties will become more and more comparable to those of a liquid droplet without cluster structure.

An estimate of the vibrational quanta to be expected on the α-particle model may be obtained, according to WHEELER [37*a*], by equating the potential energy of the quasi-elastic forces responsible for the different modes of vibration of the nucleus envisaged, taken for an amplitude roughly corresponding to the dissociation of an α-particle, to the absolute value of the binding energy of this particle to the residual nucleus, as it results from comparison of mass-defects, plus the zero-point energy of the nuclear vibrations: for the lightest α-nuclei, such an estimate yields quanta of the order of a few MeV, just as those of the lowest ($n = 2$) mode of surface vibration of the nuclear liquid drop according to (7). As regards the order of magnitude of the rotational quanta $\hbar^2/I_s$, we get for the moment of inertia I_s of a spherical homogeneous distribution of the nuclear mass

$$I_s = \tfrac{2}{5} M A R^2 = \tfrac{2}{5} M r_0^2 A^{\frac{5}{3}}, \tag{10}$$

whence

$$\hbar^2/I_s = \tfrac{5}{2} \cdot \frac{\hbar^2}{M r_0^2} \cdot A^{-\frac{5}{3}} \approx 54\, A^{-\frac{5}{3}}\, \mathrm{MeV}, \tag{11}$$

i.e. several hundred keV for the lightest α-nuclei, but only a few keV for heavy nuclei.

The additional invariance with respect to certain rotations or symmetries introduced into the nuclear system by the assumption of an α-particle structure will, especially in the case of light α-nuclei, cause a considerable reduction in the number of low-lying levels. For instance, only the wave-functions corresponding to even values of the angular momentum satisfy the requirement of Bose statistics for the two-α-particle system ^{8}Be. The states excluded in this way on symmetry grounds are nevertheless present in the spectrum of the α-particle system, but they appear at higher levels of excitation than in the spectrum of the corresponding system of nucleons without α-particle structure; for their eigen-

functions can only be given the required antisymmetry with respect to all nucleons by excitation of α-particles or subdivision into other than α-clusters. At higher excitation energies, the spectrum, as already mentioned, merges into that of a vibrating fluid continuum: this tendency toward a purely classical behaviour for high energies is just an aspect of the general correspondence argument of quantum theory.

Such an exclusion of low levels is, however, not likely to occur in the case of heavy nuclei, since these nuclei will generally not have any particularly symmetrical configuration. In view of the small value of the rotational quanta for such nuclei, this would at first sight seem to involve a serious difficulty, for the existence of states of such small excitation energies would be in flagrant contradiction to various observations (TELLER and WHEELER [38]): on the one hand, the spectra of natural α- or β-radioactive bodies show no trace of a fine structure of this order of magnitude; on the other, when β-rays are emitted by internal conversion in a transition from a metastable level to the ground state, they are never accompanied by γ-rays of slightly less energy, as should be the case if there were low-lying rotational levels which could be reached from the metastable state by radiative transitions. Somewhat more indirect, but still of great weight is the evidence derived from the β-active nuclei of very long life-times ^{40}K and ^{87}Rb (KONOPINSKI [43]) and from numerous cases of nuclear isomerism (FLÜGGE [41]); in all such phenomena, the small probability of a transition to the ground state is due to the large change of angular momentum involved, — an explanation which could not be upheld if there were competing transitions to rotational states of very low excitation. In fact, all the evidence concordantly points to an order of magnitude of a few hundred keV for the lowest excited states of even the heaviest nuclei.

The effect of symmetrical arrangement of the α-particles in pushing upward a number of levels of excitation of the system is, however, as emphasized by TELLER and WHEELER [38], supplemented by another interesting mechanism, very effective in producing the same result. Suppose that there is a rotation by which the nucleus is brought to a configuration not exactly but nearly coinciding with the initial one: a relatively small *translation* of the constituent particles will then suffice to restore the initial configuration. We have then to do with a "resonance" between two equilibrium configurations separated by a potential barrier, and if this barrier is sufficiently low for the frequency of oscillation from the one to the other configuration to be larger than the frequency of the corresponding rotation, a considerable disturbance of the rotation level results, the effect of which will be the same as that of a higher symmetry of the nucleus.

A simple example will serve to illustrate this peculiar resonance phenomenon. Consider a system of two particles attached by rigid rods of equal lengths to a pivot, around which they may rotate in the plane of the rods

and oscillate with respect to each other under the action of a potential V. Suppose there are two equilibrium positions of the system for the values $\vartheta_0(<\pi)$ and $2\pi - \vartheta_0$ of the angle ϑ between the rods: the potential $V(\vartheta)$ has then the form represented on fig. *13.15*. The possible energy eigenvalues of this system are

$$E = E_n + \frac{\hbar^2}{2I} K^2 \pm B_n, \tag{12}$$

(I moment of inertia, K rotational quantum number, E_n energy of harmonic vibration energy around the equilibrium positions, B_n energy change due to resonance between these two positions) and the corresponding eigenfunctions

$$\Psi \sim [\Psi_V(\vartheta_0) \mp \Psi_V(2\pi - \vartheta_0)]\, e^{iK\varphi} \tag{13}$$

(φ angular displacement of centre of gravity of particles). If the particles obey Bose statistics, Ψ must remain the same when the system is rotated through an angle π and brought from one equilibrium position to the other. Accordingly, the only energy values allowed in this case are

$$E = E_n + \frac{\hbar^2}{2I} K^2 - (-1)^K B_n : \tag{14}$$

the odd rotational levels are raised, the even ones depressed. As soon as the (circular) frequency $2B_n c/\hbar$ of the exchange between the two equilibrium configurations becomes larger than the rotational frequency $\hbar c K/I$, the order of the successive levels $K - 1$, K is inverted. On further increase of B_n, we shall finally find that a number of even levels will all lie below the first odd level, — a situation approaching that of the rigid system with $\vartheta_0 = \pi$, for which only the even levels subsist.

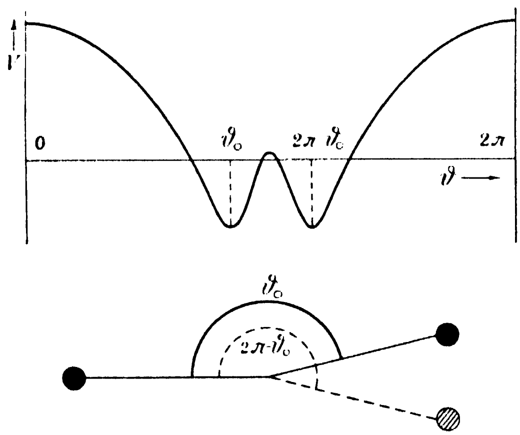

Fig. 13.15. Resonance between rotation and translation through potential barrier.

Coming back to the case we were discussing, let δ be the displacement which restores the initial configuration after it has been rotated through an angle φ; this means that the latter rotation just corresponds to a linear displacement $\approx \delta$ and therefore to an increase in kinetic energy $\approx \dfrac{\hbar^2}{2MA(\delta/\varphi)^2}$. For the purpose of a rough estimation, we may identify

this with the energy change due to the resonance; let us further take $\varphi \approx \pi$ and $\delta \approx r_\alpha$, r_α being the average half-distance between two α-particles

$$r_\alpha \approx 4^{\frac{1}{3}} r_0 \approx 1{,}6\, r_0. \tag{15}$$

The excitation energy of the state considered will thus be about *

$$\frac{\hbar^2 \pi^2}{2MAr_\alpha^2} \approx \frac{40}{A} \text{ MeV}, \tag{16}$$

i.e. $\approx$ 0,2 MeV for $A = 200$, which is indeed of the order of magnitude indicated by the empirical evidence. The quantum number J_{exc} of the first excited rotational level would accordingly be given by

$$\frac{\hbar^2 J_{\text{exc}}^2}{2I_s} \approx \frac{\hbar^2 \pi^2}{2MAr_\alpha^2},$$

or, by (10), (15),

$$J_{\text{exc}} \approx 1{,}26\, A^{\frac{1}{3}} \quad (J_{\text{exc}} \approx 8 \text{ for } A \approx 200). \tag{17}$$

Another characteristic J-value is that above which the order of the rotation levels will not be disturbed, and the nucleus thus will rotate as a rigid body; we may call this value J_{coll} the limit of "collective motion". It is defined by

$$\frac{\hbar^2}{2I_s}\left[J_{\text{coll}}(J_{\text{coll}}+1)-(J_{\text{coll}}-1)J_{\text{coll}}\right] \approx \frac{\hbar^2 \pi^2}{2MAr_\alpha^2},$$

i.e.

$$J_{\text{coll}} \approx \tfrac{1}{2} J_{\text{exc}}^2 \approx 1{,}56\, A^{\frac{2}{3}}, \tag{18}$$

— a condition just expressing (in agreement with the preceding argument) the equality of the frequencies of the rotation and exchange oscillation leading from the one to the other configuration. For $A = 200$, (18) yields $J_{\text{coll}} \approx 27$, corresponding to a rotation energy $\approx$ 2,5 MeV.

13.2. Nuclear states on α-particle model

13.21. *Stationary states of lightest α-nuclei.* For the purpose of ascertaining the possible sets of quantum numbers and the parity, as well as the approximate positions, of the lowest vibrational and rotational levels of α-nuclei, it is permissible, according to the above discussion, to treat these nuclei as simple configurations of α-particles, to which the well-known methods of molecular spectroscopy may be applied. This has been done fairly completely for the three lightest α-nuclei, ^{8}Be, ^{12}C and ^{16}O, and to some extent also for the next, ^{20}Ne. We shall give here a brief

* The same order of magnitude for the excitation energies can also be derived on the individual Fermi gas model; cf. TELLER and WHEELER [38].

review of the main results, referring the reader to the original papers for a fuller account.

Beryllium (WHEELER [37a]). The nucleus ^{8}Be is compared with a simple dumbbell configuration. There is one non-degenerate mode of vibration; the rotation energy has the form $(\hbar^2/2I_c)J(J+1)$, I_c being the moment of inertia around a symmetry axis perpendicular to the line joining the centres of the α-clusters. Only the levels with even J are allowed; they are of even parity.

Carbon (WHEELER [37a]). The equilateral triangle configuration representing the ^{12}C nucleus has two distinct fundamental modes of vibration: a non-degenerate mode, corresponding to an isotropic dilatation and contraction of the triangle (quantum number n_1), and a doubly degenerate mode, describable as a tipping * of the triangle (quantum number n_2). The rotation is that of a symmetrical top, with energy

$$\frac{\hbar^2}{2I_c}J(J+1)+\frac{\hbar^2}{2}\left(\frac{1}{I_a}-\frac{1}{I_c}\right)\Lambda^2; \tag{1}$$

the moments of inertia I_a, I_c refer to the symmetry axis perpendicular to the plane of the triangle (figure axis), and to a symmetry axis in this plane, respectively; Λ denotes the quantum number of the projection of the angular momentum on the figure axis, an integer limited by the condition $|\Lambda| \leq J$.

The parity of the allowed levels is given by the following table; it is characterized by a parameter p, which has the value $+1$ or -1 according as the level is even or odd. The notation ± 1 means a pair of coinciding levels, one of them even, the other odd; and $n(\pm 1)$ denotes n such pairs.

13.21. *Parity of levels of* ^{12}C

$\lvert\Lambda\rvert$	J	$n_1 = 0, 1, 2\ldots;\ n_2 =$ 0	1	$2n$	$2n+1$
0	even	$+1$	·	$+1$	$+1$
	odd	·	·	·	-1
3λ	any	$(-1)^{\Lambda}$	·	$(-1)^{\Lambda}$	$+1$ -1
$3\lambda \pm 1$	any	·	± 1	$n(\pm 1)$	$n(\pm 1)$

Oxygen (WHEELER [37a], DENNISON [40]). The regular tetrahedron, which represents the ^{16}O nucleus, has 3 distinct fundamental modes of vibration: a non-degenerate isotropic dilatation and contraction (n_1), a doubly degenerate motion, in which the α-particles are paired into two dumbbells twisting with respect to each other (n_2), and a trebly degenerate

* In this mode of vibration, one side of the triangle shrinks and at the same time moves away from the third α-particle.

vibration, in which one dumbbell shortens while the other lengthens (n_3). The rotation is that of a spherical top, with energy $(\hbar/^2I_s)J(J+1)$, I_s being defined by (*13.15*–10). Besides the sum of rotation and vibration energy, and the usual interactions between rotation and vibration (which, for ^{16}O, would be as large as 0,5 ... 1 MeV), two other perturbations must be taken into account. One correction arises from the fact that the third mode of vibration has an angular momentum; Dennison shows, e.g., that for the state $n_1 = n_2 = 0$, $n_3 = 1$, $J = 1$, it amounts to $\frac{9}{8}\frac{\hbar^2}{I_s}$. The other correction is due to the effect, discussed in *13.15*, of exchange between two inverse configurations. In the case of ^{16}O, the inversion may be produced by a vibration of the third kind, and the additional energy term has the form $\mp G_{n_3}$ with the upper or lower sign according as the wave-function of the state remains invariant or changes sign when the inversion is performed; Dennison estimates that for $n_3 = 0$, G_0 is of the order of $1{,}5 \cdot 10^{-3}$ times the quantum of vibration of the third mode, and for $n_3 = 1$, $G_1 \approx 25\, G_0$.

Neon (TELLER and WHEELER [38]). When no vibration is excited, the bipyramid configuration of ^{20}Ne behaves much like ^{12}C: the rotation energy has the form (1) and the allowed levels are given, with their parity, by the $n_2 = 0$ column of table *13.21*. There is an additional energy term due to the exchange between two resonating configurations (fig. *13.21*).

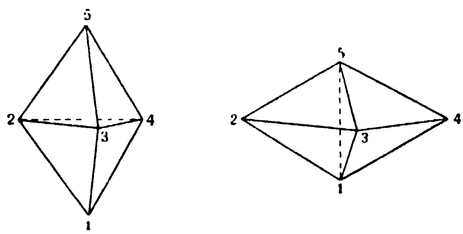

Fig. 13.21. The two resonating configurations of the ^{20}Ne nucleus.

It has been evaluated by Teller and Wheeler for the states without vibrational excitation; they find

$$-\frac{3}{2^J}\,\varepsilon\, G \sum_{f=0}^{J-\Lambda}(-1)^f \binom{J+\Lambda}{f}\binom{J-\Lambda}{f} \tag{2}$$
$$(\varepsilon = 2 \text{ for } \Lambda \neq 0 \quad , \quad \varepsilon = 1 \text{ for } \Lambda = 0),$$

and give an estimate, probably too high, of $G \approx 0{,}16$ MeV.

13.211. *Comparison with experiment.* Although a number of excited states of the nuclei just discussed have been disclosed by the study of nuclear reactions, and their excitation energies determined, the comparison of these results with the theoretical predictions is rendered difficult by the deficient information on the parities or angular momenta of these states. Only in the case of ^{16}O could the properties of the excited states formed in the reaction $^{19}_{9}F(p, \alpha)^{16}_{8}O$ be analyzed in a sufficiently com-

plete way (OPPENHEIMER and SCHWINGER [39]). Starting from these data, DENNISON [40] was led to definite assignments of the vibration frequencies and the moment of inertia of the nucleus; the latter is found to be rather smaller than would result from formula (*13.15*–11).

The moments of inertia of the other nuclei can also be estimated, provided a plausible assumption is made in each case about the ratio of the axes of the spheroid of inertia. In fact, by comparing the moments of inertia of a spherical and a spheroidal mass distribution of equal volumes, it is easily proved that

$$\frac{\hbar^2}{2\,I_a} = \xi^{\frac{2}{3}} \frac{\hbar^2}{2\,I_s}, \qquad \frac{\hbar^2}{2\,I_c} = \frac{2\,\xi^{\frac{2}{3}}}{1+\xi^2}\,\frac{\hbar^2}{2\,I_s}, \tag{3}$$

ξ denoting the ratio of polar to equatorial diameter of the spheroidal mass distribution. The following table gives the results pertaining to the four

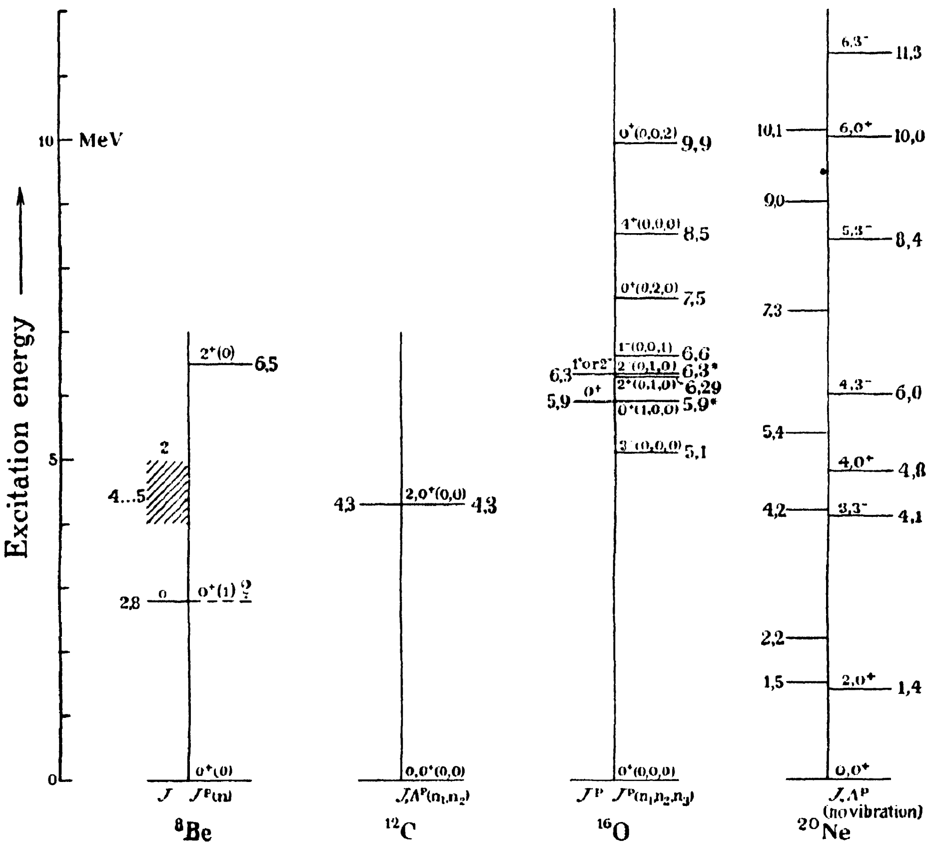

Fig. 13.211. Excited states of α-nuclei.

The empirical data are represented on the left, the theoretical spectrum on the right (all excitation energies being given in MeV). Each state is characterized by its quantum numbers, as indicated below the corresponding spectrum (p denoting the parity factor, $+1$ for even states, -1 for odd states). The ^{16}O levels marked with an asterisk are those on which Dennison's analysis is based.

lightest α-nuclei; the ξ-values are those proposed by WHEELER [37a] and TELLER and WHEELER [38]. The values of $\hbar^2/2I_s$ calculated by (*13.15*–11) have been multiplied by a correction factor 1,6 in conformity with Dennison's analysis of the ^{16}O spectrum; there is, of course, much uncertainty both in this correction and in the choice of the ξ's.

13.211. *Rotation parameters of light α-nuclei*

Nucleus	ξ	$\frac{\hbar^2}{2I_s}$	$\frac{\hbar^2}{2I_c}$	$\frac{\hbar^2}{2I_a} - \frac{\hbar^2}{2I_c}$
^8Be	1,5	1,34	1,07	—
^{12}C	0,67	0,69	0,74	− 0,21
^{16}O	1	0,43	0,43	0
^{20}Ne	1,46	0,30	0,24	0,14

Energies in MeV

The degree of agreement between the computed spectra and the empirical evidence may be judged from fig. *13.211*; as regards the three lightest nuclei, this diagram embodies the results of Dennison's discussion (modified as regards ^8Be in conformity with *13.11*). The theoretical ^{20}Ne-spectrum is restricted to the rotation levels, without vibrational excitation; the resonance correction (2) has been omitted, since it is presumably very small. The experimental data concerning ^{20}Ne and heavier α-nuclei are collected and critically discussed in a paper by GUTHRIE and SACHS [42]; it would be premature in those cases to attempt any theoretical identification of the observed states of excitation.

13.22. *Stationary states of nuclei of mass number $4a \pm 1$.* The symmetry and invariance considerations which led to the above results for the light α-nuclei have been extended by HAFSTAD and TELLER [38] to the neighbouring nuclei * of mass number $4a \pm 1$, considered as aggregates of α-clusters and an additional loose nucleon or an α-cluster lacking one nucleon (cluster with a "hole"). As long as electrostatic forces are neglected, it does not matter whether the loose nucleon or hole is a neutron or a proton **. The method is based on the observation that, owing to the short range of the nuclear forces, a loose nucleon will interact strongly with only one α-particle at a time: it will therefore be possible to express its individual wave-function as a linear combination of wave-functions representing the loose nucleon in the neighbourhood of each separate α-particle. The latter wave-functions have each a node through the centre of gravity of the respective α-particle; the state of lowest energy of the system of mass number $4a + 1$ will correspond to the particular linear

* The treatment of mass number 19 is sketched by SACHS [39].

** An estimate of Coulomb energies on the present model (leading to less satisfactory results than on the individual models (*3.3*)) has been carried out by BROWN and INGLIS [39].

combination (of the required symmetry) which has the least number of nodes. In the case of mass number $4a - 1$, the wave-function of the hole will be built up quite similarly, but in order to get the smallest energy, we have to choose that with the largest number of nodes. A closer discussion of symmetry properties of such functions for the different mass number values leads then without difficulty to an enumeration of the lowest excited levels in each case; the rotation angular momentum must here, in conformity with spectroscopic use, be denoted by K instead of J, the latter notation being reserved for the total angular momentum, including spin, $J = K \pm \frac{1}{2}$.

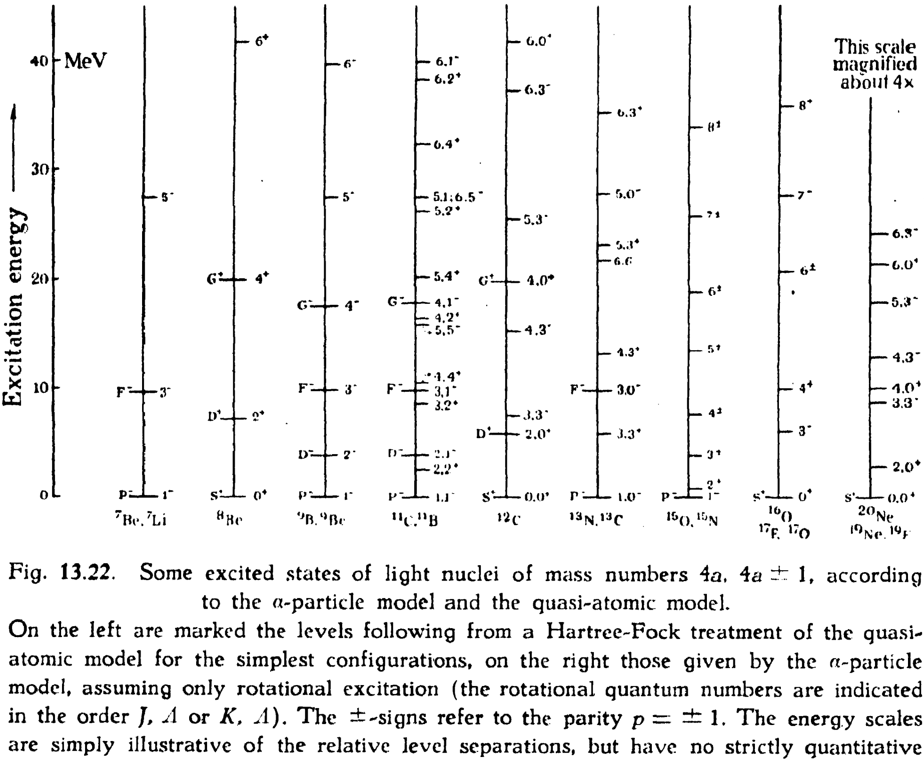

Fig. 13.22. Some excited states of light nuclei of mass numbers $4a$, $4a \pm 1$, according to the α-particle model and the quasi-atomic model.
On the left are marked the levels following from a Hartree-Fock treatment of the quasi-atomic model for the simplest configurations, on the right those given by the α-particle model, assuming only rotational excitation (the rotational quantum numbers are indicated in the order J, Λ or K, Λ). The $\pm$-signs refer to the parity $p = \pm 1$. The energy scales are simply illustrative of the relative level separations, but have no strictly quantitative significance; they have been fitted to each other by arbitrarily identifying the calculated energies of the corresponding levels $D+$, $2+$ of ^{8}Be.

Fig. *13.22*, reproducing with a slight extension a diagram in Hafstad and Teller's paper, conveniently summarizes these results and parallels them with the analogous predictions following from the quasi-atomic model (*10.2*). It will be seen that both models are in complete agreement as regards angular momentum and parity of the ground state and also for excited states exhibit a striking similarity in these respects. In judging this resemblance, one should not forget that the spectra represented are very incomplete: on the quasi-atomic model, only the levels corresponding to the simplest configurations have been investigated; on the α-particle model,

the vibrational excitation has been entirely left out, and no account has been taken of levels corresponding to other wave-functions of the loose nucleon (or hole) than that with the smallest (or largest) number of nodes *.

Hafstad and Teller also show how a rough estimate of the mass-defects of the nuclei of the class under discussion can be derived from their treatment; for this aspect of the problem, however, it will suffice to refer the reader to their paper.

13.3. The resonating group structure

The prominent part played by α-clusters in nuclear structure is due to their peculiar stability. But other clusterings, in fact all conceivable groupings of the constituent nucleons into clusters, must clearly also occur in the course of time, only with widely varying probabilities. The consideration of such alternative groupings is especially important for the discussion of nuclear reactions and scattering processes, but their virtual existence should also be taken into account in any attempt to improve calculations of energies of stationary states. One is here reminded of the situation met with in the study of molecular binding energies: in many cases, a "resonance" between different types of binding (e.g. homopolar and heteropolar) in a molecule is found to have considerable influence on its energetic and spectroscopic properties. Pursuing the fruitful analogy between nuclear and molecular structures, we should likewise, as emphasized by WHEELER [37a], regard any state of a nucleus as a superposition of all possible kinds of nucleon clusters, i.e. (as Wheeler puts it) attribute to the nucleus a *"resonating group structure"*. The corresponding eigenfunction should accordingly be set up as a linear combination of products of wave-functions representing the various "configurations" or groupings into clusters; the coefficients, depending on the coordinates of the centres of gravity of the clusters, should be determined as proper solutions of a certain set of equations involving these coordinates as variables. This method, which thus provides a much broader and more rational basis than the α-particle model for an attack on the properties of complex nuclei, has been carefully discussed, and its general algorithm very clearly set out, in a fundamental paper by WHEELER [37b]. To this paper the reader is referred for a detailed study; we shall here only give a brief outline of the main points of interest.

Let us, for the sake of illustration, take an especially simple case. Retaining only the most important ** configurations occurring in a given

* Cf. on this point the case of ^{7}Li, discussed in *17.52*.

** The configurations which are liable to contribute most effectively to the building up of a particular nuclear state can readily be determined by a group-theory argument due to WHEELER [37a]. As long as we neglect the spin dependence of the nuclear forces, any (degenerate) state of a nucleus of mass number A belongs to a definite irreducible

nuclear state, let us suppose that they all consist of only two clusters. The total eigenfunction is then of the form

$$\Psi \sim \sum_i \overbrace{F_i\,[\vec{X}_i(\mathrm{I}), m_{s_i}(\mathrm{I});\ \vec{X}_i(\mathrm{II}), m_{s_i}(\mathrm{II})]\ \Phi_i(\mathrm{I})\ \Phi_i(\mathrm{II})}, \tag{1}$$

the summation extending over all configurations, distinguished by an index i; $\Phi_i(\mathrm{I})$, $\Phi_i(\mathrm{II})$ are the wave-functions of the two clusters I, II in configuration i, $\vec{X}_i(\mathrm{I})$, $\vec{X}_i(\mathrm{II})$ the radii vectores of their centres of gravity and $m_{s_i}(\mathrm{I})$, $m_{s_i}(\mathrm{II})$ their spin quantum numbers (expressing the spin components in an arbitrarily chosen direction). The brace over the typical term in the sum (1) means that it must be replaced by the antisymmetric combination of all similar expressions derived from it by permuting the nucleon variables. The coefficients F_i, which depend on the $\vec{X}$'s only through the differences $\vec{X}_i(\mathrm{I})-\vec{X}_i(\mathrm{II})$, may conveniently be written as symbolical vectors $\boldsymbol{F}_i$, the components of which are numbered by the spin variables $m_{s_i}(\mathrm{I})$, $m_{s_i}(\mathrm{II})$. It can then be shown that for *any* type of nuclear interaction (even including many-body forces) these $\boldsymbol{F}_i$ must satisfy a set of integro-differential equations of the type

$$\begin{aligned}&\left[\frac{\hbar^2}{2M_i}\Delta_{\vec{X}} + \mathcal{E} - \mathcal{E}_i\right] \boldsymbol{F}_i(\vec{X}) \\ &\qquad = \sum_j \{ \boldsymbol{V}_{ij}(\vec{X})\, \boldsymbol{F}_j(\vec{X}) + \int [\boldsymbol{J}_{ij}(\vec{X},\vec{\xi}) - \mathcal{E}\, \boldsymbol{I}_{ij}(\vec{X},\vec{\xi})]\, \boldsymbol{F}_j(\vec{\xi})\, dv_{\xi}\};\end{aligned} \tag{2}$$

here M_i denotes the reduced mass of the two clusters in configuration i, $\mathcal{E}_i$ the total energy of these separated clusters and $\mathcal{E}$ the energy of the nuclear state considered. The operators $\boldsymbol{V}_{ij}$, $\boldsymbol{J}_{ij}$, $\boldsymbol{I}_{ij}$ can be calculated in terms of the law of nuclear interaction and the cluster wave-functions Φ; for $i = j$, they represent the interactions between the clusters in the same configuration; for $i \neq j$, the resonance effects between different configurations (in particular, the transmutations) *. The integral in (2) arises from

representation D_A of the symmetric group comprising all permutations of A elements (*10.31*). If now two clusters of N_1, N_2 nucleons in states characterized by the respective representations D_{N_1}, D'_{N_2} are combined into a nucleus, the states of the latter arising from this combination will directly be found by resolving the direct product $D_{N_1} \times D'_{N_2}$ into its irreducible components. Since further the energies of any nuclear state can be approximately evaluated in terms of the corresponding representation (*10.31*), we can in each case roughly compare the energy of the state of the compound nucleus in which we are interested with the total energies of the groups of separated clusters which may occur in its formation: we have then to retain only those groups of clusters for which these two energy values are not too different from each other.

* It may be important to take account of the polarization of the clusters due to their mutual interaction. In principle, such polarization effects can be included in the general formulae (1) and (2) in either of two equivalent and mutually exclusive ways, viz. assuming the cluster wave-functions also to depend in an appropriate manner on the $\vec{X}$'s, or introducing suitably chosen configurations in which excited states of the clusters occur.

exchange terms of potential energy as well as from the lack of orthogonality of the cluster wave-functions; it can be interpreted (*11.121*) as expressing a velocity dependence of the cluster interactions *, — a feature not to be confused, of course, with a possible velocity dependence of the elementary interactions between nucleons.

The next step consists in transforming the system (2) to polar coordinates, thereby reducing it to a system of the same form, but containing only unknown functions of the radial coordinate. It then becomes possible to transform it further to a system of linear integral equations, the eigenvalues $\mathcal{E}$ of which are determined by equating to zero its Fredholm determinant. This demands, of course, that all $\mathcal{E}_i$ be larger than $\mathcal{E}$, for only then do we have a completely stable system. If this condition is not fulfilled (which implies the occurrence of scattering or transmutation processes), the Fredholm determinant may be made $= 0$ for any value of $\mathcal{E}$, because it contains the undetermined phases introduced by taking account of the required asymptotic behaviour of the solutions. Still, as Wheeler shows, the equation Fredholm determinant $= 0$ can be used in this case to determine almost completely the cross-sections of the scattering or transmutation processes (it has, however, to be supplemented by some other procedure in order to resolve the remaining ambiguities for one arbitrary energy value).

Up to now, the method of resonating group structure has only been applied to the simplest nuclear systems, especially the 3-nucleon systems and ^{5}He; these applications, briefly discussed in the following Chapter, will serve to illustrate the general procedure **. However, in these instances, the last step, viz. the solution of the radial integro-differential equations, has been performed by special methods, involving certain approximations. The results are rather too inconclusive to enable one to form a judgment on the merits of the resonating group method for purposes of practical calculations. It is clear that its complication rapidly becomes overwhelming as the number of configurations is increased in order to improve the accuracy. Nevertheless, considering its extreme generality and the soundness of its theoretical foundations, this method fully deserves attention as the one most liable to lead to essential advance in the problem of the constitution of complex nuclei, the difficulty of which has been so often displayed in the preceding sections.

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* This general property of cluster interactions has been known for a long time in the special case of the force exerted by a nucleus on one of its constituent nucleons ("van Vleck's potential"). Cf. B & B, § 37.

** Applications of the method to the 4-nucleon (α-particle) system by PARKER [37] and the system of two α-particles by WHEELER [37*a*] have been announced, but not published. For its application to ^{7}Li, see KITTEL [42].

CHAPTER XIV

PROPERTIES OF LIGHT NUCLEI

14.0. The methods of treatment of complex nuclei, discussed in the preceding Chapters, have also been applied, with varying degrees of success, to the investigation of the lightest nuclei consisting of more than two nucleons. Especially the systems of 3 or 4 nucleons have been studied in a rather detailed way, either by the Hartree-Fock variational method (*9.21*) or by the more refined resonating group method (*13.3*); we shall in this Chapter confine ourselves to these systems and give a brief account of the present state of their theoretical and experimental investigation. Since the interaction between two neutrons cannot be studied directly from phenomena involving only two such particles, the 3-nucleon systems are the simplest ones liable to yield indirect information about this interaction; unfortunately, the issue is obscured by the possible occurrence of many-body forces (*1.34*). The following discussion will be based on the assumption of central interactions between pairs of nucleons only; within this limitation, it will appear that important indications as regards the charge dependence of the nuclear interaction can in fact be derived from the analysis of the scattering of nucleons by deuterons.

14.1. The three-nucleon systems

14.11. *Stationary states on resonating group method.* According to the conception of resonating group structure, a system of 3 nucleons (not all of them protons or all neutrons) has to be represented as a combination of a deuteron and a loose nucleon (neutron or proton). Clearly, this representation is especially suited for the treatment of scattering problems; but it can just as well be used for describing the stationary states of binding of the nuclei ^3H and ^3He. Its most serious drawback is its inability to take account easily of the polarization of the deuteron in the field of the third nucleon. The total wave-function is written as a symmetrical combination

$$\Psi(1,2,3) = \psi(\widetilde{12},3) + \psi(\widetilde{23},1) + \psi(\widetilde{31},2) \tag{1}$$

of wave-functions such as $\psi(\widetilde{12},3)$, antisymmetrized with respect to the pair (1,2), constituting the deuteron. The function $\psi(\widetilde{12},3)$ is further expressed as a product of three factors:

$$\psi(\widetilde{12},3) = \varphi(\overline{12},3)\,\sigma(\overline{12},3)\,\tau(\widetilde{12},3), \tag{2}$$

containing the spatial, spin and isotopic coordinates, respectively. If the

deuteron group is assumed to be in the ground state, the isotopic factor,

$$\tau(\widetilde{12},3) = {}^1(\tau)_0(1,2)\, u_\pm(3), \tag{3}$$

is antisymmetric in 1 and 2, while the spin factor, involving ${}^3(\sigma)(1,2)$, is symmetrical in these particles; consequently the space factor is also symmetrical.

The stationary states can be classified, according to the spin multiplicity, into a doublet and a quartet system. The respective spin eigenfunctions are readily found to be:

Doublet system:

$$\begin{aligned}
{}^2\sigma_{\frac{1}{2}}(\overline{12},3) &= \frac{1}{\sqrt{3}}\,[{}^3(\sigma)_0(1,2)\, v_+(3) - \sqrt{2}\cdot {}^3(\sigma)_1(1,2)\, v_-(3)] \\
{}^2\sigma_{-\frac{1}{2}}(\overline{12},3) &= \frac{1}{\sqrt{3}}\,[{}^3(\sigma)_0(1,2)\, v_-(3) - \sqrt{2}\cdot {}^3(\sigma)_{-1}(1,2)\, v_+(3)]
\end{aligned} \tag{4}$$

Quartet system:

$$\begin{aligned}
{}^4\sigma_{\frac{3}{2}}(\overline{12},3) &= {}^3(\sigma)_1(1,2)\, v_+(3) \\
{}^4\sigma_{\frac{1}{2}}(\overline{12},3) &= \frac{1}{\sqrt{3}}\,[\sqrt{2}\cdot {}^3(\sigma)_0(1,2)\, v_+(3) + {}^3(\sigma)_1(1,2)\, v_-(3)] \\
{}^4\sigma_{-\frac{1}{2}}(\overline{12},3) &= \frac{1}{\sqrt{3}}\,[\sqrt{2}\cdot {}^3(\sigma)_0(1,2)\, v_-(3) + {}^3(\sigma)_{-1}(1,2)\, v_+(3)] \\
{}^4\sigma_{-\frac{3}{2}}(\overline{12},3) &= {}^3(\sigma)_{-1}(1,2)\, v_-(3).
\end{aligned} \tag{5}$$

The spatial dependence of the wave-function is further split into two factors:

$$\varphi(\overline{12},3) = \chi(|\vec{x}^{(1)} - \vec{x}^{(2)}|)\, F\,[\vec{x}^{(3)} - \tfrac{1}{2}(\vec{x}^{(1)} + \vec{x}^{(2)})]; \tag{6}$$

in this formula, χ is the radial eigenfunction of the deuteron, F the probability amplitude pertaining to the position of the third nucleon with respect to the centre of gravity of the deuteron. This last function F is the only one which is not specified beforehand. It has to satisfy a certain integro-differential equation, which is obtained by inserting into the wave-equation for Ψ its expression (1) with the choice described above of the various factors of the φ's. This equation for F has been first given by WHEELER [37*a*] for the case in which there is only a proton-neutron interaction, and again derived for the most general form of central nuclear potential by BUCKINGHAM and MASSEY [41] and by HÖCKER [42]. We shall not reproduce the details of the derivation, which may best be looked up in Buckingham and Massey's paper *. The resulting form of the

* There is a strange error in this otherwise excellent paper. The authors give two separate treatments of the doublet states, (*a*) by using isotopic coordinates, (*b*) by treating

fundamental equation of the resonating group treatment of the 3-nucleon problem is

$$\left\{\frac{3\hbar^2}{4M}\Delta+\varepsilon'-\alpha\,U(r)-\eta\,C(r)\right\}F(\vec{x})=$$
$$(\tfrac{4}{3})^3\int\{\beta\,Q(\vec{x},\vec{x}')+\gamma\,[P(\vec{x},\vec{x}')+(\varepsilon'-\tfrac{5}{8}\varepsilon_0)\,N(\vec{x},\vec{x}')]\}\,F(\vec{x}')\,dv'. \tag{7}$$

Here, $\varepsilon'=\varepsilon-\varepsilon_0$ denotes the difference between the total energy ε of the system and the binding energy ε_0 of the deuteron; in a state of relative motion of the third nucleon with respect to the deuteron, ε' is the kinetic energy of this motion, equal to 2/3 the kinetic energy of the nucleon in the laboratory system (in which the deuteron is at rest). The other symbols are defined as follows: putting

$$\vec{u}=\tfrac{2}{3}(\vec{x}+2\vec{x}'),\qquad \vec{v}=\tfrac{2}{3}(\vec{x}'+2\vec{x}), \tag{8}$$

and adopting the form (*8.2*–1 or 4) for the nuclear potential, — so that $J(r)$ represents the absolute value of the effective potential in the ground state of the deuteron, — one has

$$\begin{aligned}
U(r)&=\int|\chi(r')|^2\,J(|\vec{x}+\tfrac{1}{2}\vec{x}'|)\,dv'\\
C(r)&=\int|\chi(r')|^2\frac{e^2}{|\vec{x}+\tfrac{1}{2}\vec{x}'|}\,dv'\\
Q(\vec{x},\vec{x}')&\equiv\chi(u)\,J(|\vec{u}-\vec{v}|)\,\chi(v)\\
P(\vec{x},\vec{x}')&=-\chi(u)\left[\tfrac{1}{3}(J(u)+J(v))+\eta\frac{e^2}{|\vec{u}-\vec{v}|}\right]\chi(v)+\frac{4\hbar^2}{3M}\operatorname{grad}_{\vec{u}}\chi(u)\cdot\operatorname{grad}_{\vec{u}}\chi(v)\\
N(\vec{x},\vec{x}')&=\chi(u)\,\chi(v).
\end{aligned} \tag{9}$$

The constants α, β, γ are linear combinations of the interaction parameters, different for doublet and quartet states:

Doublet:	*Quartet:*	
$\alpha=2W-B-H+\tfrac{1}{2}M$	$\alpha=2W+2B-H-M$	
$\beta=\tfrac{1}{2}W-B-H+2M$	$\beta=-W-B+2H+2M$	(10)
$\gamma=-\tfrac{1}{2}$	$\gamma=1,$	

or, by (*8.2*–5),

$\alpha=2(a_0-2a_\sigma)$	$\alpha=2(a_0+a_\sigma)$	
$\beta=\tfrac{1}{2}[a_0-5a_\sigma+3(a_\tau-5a_{\sigma\tau})]$	$\beta=-[a_0+a_\sigma+3(a_\tau+a_{\sigma\tau})]$	(11)
$\gamma=-\tfrac{1}{2}$	$\gamma=1.$	

protons and neutrons as distinct particles; they seem to expect — and actually find — different results in the two cases, whereas both treatments are of course equivalent (*4.14*). In case (*a*), however, they use a wrong spin function (which actually represents a mixture of doublet and quartet states). Everything that concerns case (*a*) in their paper should therefore be omitted; their case (*b*) contains the correct results.

The constant η is, of course, 0 or 1 according as the third nucleon is a neutron or a proton.

14.111. *Doublet states on individual model.* The wave-function of a doublet state appears, on the individual model, in a somewhat different form. On this model, it is in fact natural to group the two like nucleons together. If we write the total wave-function in the form, analogous to (1),

$$\Psi(1,2,3) = \psi'_{\mathrm{ind}}(\widetilde{12},3) + \psi_{\mathrm{ind}}(\widetilde{23},1) + \psi'_{\mathrm{ind}}(\widetilde{31},2), \tag{12}$$

the isotopic factor of the function $\psi'_{\mathrm{ind}}\ (\widetilde{12},3)$ will be

$$\tau(\overline{12},3) = {}^{3}(\tau)_{\pm 1}(1,2)\, u_{\mp}(3), \tag{13}$$

and we are primarily concerned with the antisymmetrical spin states

$$^{2}\sigma_{\pm\frac{1}{2}}(\widetilde{12},3) = {}^{1}(\sigma)_{0}(1,2)\, v_{\pm}(3). \tag{14}$$

However, the symmetrical states $^{2}\sigma_{\pm\frac{1}{2}}(\overline{12},3)$ will also occur to some extent; the function $\psi'_{\mathrm{ind}}(\widetilde{12},3)$ will be a linear combination of the type

$$\psi'_{\mathrm{ind}}(\widetilde{12},3) = [\varphi_{\mathrm{ind}}(\overline{12},3)\cdot {}^{2}\sigma(\widetilde{12},3) \ + \ \varphi_{\mathrm{ind}}(\widetilde{12},3)\cdot {}^{2}\sigma(\overline{12},3)]\cdot \tau(\overline{12},3). \tag{15}$$

Formally, of course, our resonating group solution (1), (2), (3), (4) can be put into the individual model form (12), (13), (14), (15); one has just to take

$$\begin{aligned} \varphi'_{\mathrm{ind}}(\overline{12},3) &= \frac{3}{\sqrt{2}}\,[\varphi(31,2) + \varphi(\overline{23},1)] \\ \varphi'_{\mathrm{ind}}(\widetilde{12},3) &= \frac{1}{\sqrt{2}}\,[\varphi(\overline{31},2) - \varphi(23,1)]. \end{aligned} \tag{16}$$

But the procedure generally followed when treating the states of binding of ^{3}H or ^{3}He on the individual model consists in adopting for the $\varphi'_{\mathrm{ind}}\ (\overline{12},3)$ and $\varphi_{\mathrm{ind}}\ (\widetilde{12},3)$ some simple analytical form involving a small number of parameters, which are then varied so as to minimize the expression for the total energy of the system.

After carrying out the summations over spin and isotopic coordinates, this last quantity (in the case of ^{3}H) may be written

$$\varepsilon = \frac{1}{N_s + N_a}\Big[-\frac{\hbar^2}{2M}(K_s + K_a) + a_s J_s + a_a J_a + a'_s J'_s + a'_a J'_a \\ + a^{\mathrm{ex}}_s J^{\mathrm{ex}}_s + a^{\mathrm{ex}}_a J^{\mathrm{ex}}_a \mp a_\times J_\times \mp a^{\mathrm{ex}}_\times J^{\mathrm{ex}}_\times\Big], \tag{17}$$

with the following notation: the N's are the normalization integrals

$$N_s = \int [\varphi(\overline{12},3)]^2\, dv^{(1)}\, dv^{(2)}\, dv^{(3)} \quad , \quad N_a = \int [\varphi(\widetilde{12},3)]^2\, dv^{(1)}\, dv^{(2)}\, dv^{(3)}; \tag{17a}$$

the K's pertain to the kinetic energy, the J's to the potential energy, the

indices s and a signifying that the functions $\varphi(\overline{12},3)$ or $\varphi(\widetilde{12},3)$, respectively, enter into their definition:

$$\begin{aligned} K &= \int \varphi(12,3)\,(\Delta^{(1)}+\Delta^{(2)}+\Delta^{(3)})\,\varphi(12,3)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)} \\ J &= \int \varphi(12,3)\,\mathcal{J}(12)\,\varphi(12,3)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)} \\ J' &= \int \varphi(12,3)\,\mathcal{J}(23)\,\varphi(12,3)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)} \\ J^{\mathrm{ex}} &= \int \varphi(12,3)\,\mathcal{J}(23)\,\varphi(31,2)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)}; \end{aligned} \tag{17b}$$

finally,

$$\begin{aligned} J_{\times} &= \int \varphi(\overline{12},3)\,\mathcal{J}(23)\,\varphi(\widetilde{12},3)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)} \\ J'_{\times} &= \int \varphi(\overline{12},3)\,\mathcal{J}(23)\,\varphi(\widetilde{31},2)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)}. \end{aligned} \tag{17c}$$

The double signs in (17) refer to the substates with magnetic quantum numbers $\pm\frac{1}{2}$. The numerical coefficients a can be expressed in the following equivalent ways in terms of the interaction parameters (8.2–5, 9, 12):

$$\begin{aligned} a_s &= W-B-H+M = a_0-3a_\sigma+a_\tau-3a_{\sigma\tau} &&= -q \\ a_a &= W+B-H-M = a_0+a_\sigma+a_\tau+a_{\sigma\tau} &&= -{}^3p \\ a'_s &= 2W+B = 2(a_0-a_\tau) &&= -\tfrac{1}{4}[3+q+3\,{}^3p+{}^1p] \\ a'_a &= 2W-B = 2(a_0-2a_\sigma-a_\tau+2a_{\sigma\tau}) &&= -\tfrac{1}{4}[1+3q+{}^3p+3\,{}^1p] \\ a^{\mathrm{ex}}_s &= H+2M = -2(a_\tau+3a_{\sigma\tau}) &&= -\tfrac{1}{4}[3+q-3\,{}^3p-{}^1p] \\ a^{\mathrm{ex}}_a &= -H+2M = 2(a_\tau-5a_{\sigma\tau}) &&= -\tfrac{1}{4}[1+3q-{}^3p-3\,{}^1p] \\ a_{\times} &= \sqrt{3}\,B = 2\sqrt{3}\,(a_\sigma-a_{\sigma\tau}) &&= -\tfrac{\sqrt{3}}{4}[1-q+{}^3p-{}^1p] \\ a^{\mathrm{ex}}_{\times} &= \sqrt{3}\,H = -2\sqrt{3}\,(a_\tau-a_{\sigma\tau}) &&= -\tfrac{\sqrt{3}}{4}[1-q-{}^3p+{}^1p]. \end{aligned} \tag{17d}$$

Once the ^{3}H-problem has been solved, the energy of ^{3}He is easily obtained by adding the expectation value of the Coulomb interaction between the protons:

$$\varepsilon(^3\mathrm{He}) = \varepsilon(^3\mathrm{H}) + \frac{C_s+C_a}{N_s+N_a}, \qquad C = \int \varphi(12,3)\,\frac{e^2}{r^{(12)}}\,\varphi(12,3)\,dv^{(1)}\,dv^{(2)}\,dv^{(3)}. \tag{18}$$

Calculations performed according to this scheme (BROWN [39]) have shown that the amount of symmetrical spin state in (15) is relatively small; this affords some justification for the simplified procedure adopted in most treatments of the problem, viz. the complete neglect of $\varphi(\widetilde{12},3)$, so that all integrals with indices a and $\times$ disappear from equation (17). If, moreover, one chooses for $\varphi(\overline{12},3)$ a form completely symmetrical in all pairs of nucleons, the remaining J_s, J'_s and J^{ex}_s become equal, and on account of (17d) the problem further reduces to minimizing the expression

$$\varepsilon = \frac{1}{N}\left[-\frac{3\hbar^2}{2M}K' - 3\,\frac{1+q}{2}\,J\right], \quad K' = \int \varphi(\overline{123})\,\Delta^{(1)}\,\varphi(\overline{123})\,dv^{(1)}\,dv^{(2)}\,dv^{(3)}. \tag{19}$$

It will be better to postpone the discussion of the results obtained for ^3H and ^3He by the above methods until we are in a position to compare them with those pertaining to the ^4He and ^5He nuclei. We shall now pass to the consideration of various reactions involving three nucleons.

14.12. *Scattering of neutrons by deuterons.* The total cross-section for elastic scattering of neutrons by deuterons has often been determined together with that of the proton (and other nuclei), and by the same methods. The data are collected in table *14.12.* The measurements for thermal neutrons, which have been performed with heavy water as

14.12. *Scattering cross-section of deuteron for neutrons*

Reference	Neutron energy MeV	Deuteron scattering cross-section 10^{-24} cm^2
GOLDHABER and BRIGGS [37]	thermal (300° K)	4,0 ± 0,8
POWERS *et al.* [38]	id. (300° K)	5,3 ± 0,2
	id. (120° K)	6,5 ± 0,3
BEYER and WHITAKER [40]	id. (300° K)	6,95 ± 0,2
CARROLL [41]	id. (300° K)	5,7
MARSHALL [46]	id. (300° K)	3,1
KIKUCHI and AOKI [39a]	2,45 ± 0,05	1,98 ± 0,10
AOKI [39]	2,21	2,25 ± 0,10
	2,48	2,19 ± 0,10
	2,76	2,10 ± 0,10
ZINN, SEELY and COHEN [39]	2,85 ± 0,04	2,17 ± 0,08
AMALDI *et al.* [43]	4,1	1,79 ± 0,08
	12,5	0,78 ± 0,12*
	13,5	0,89 ± 0,07*
	14	0,86 ± 0,03*
NUCKOLLS, BAILEY *et al.* [46]	0,35	3,53 ± 0,32
	0,46	3,32 ± 0,16
	0,72	3,46 ± 0,13
	0,97	2,97 ± 0,14
	1,0 ± 0,1	3,11 ± 0,20
	1,6	2,90 ± 0,12
	2,0	2,60 ± 0,08
	2,6	2,34 ± 0,06
	3,0	2,24 ± 0,09
	3,5	2,04 ± 0,07
	4,0	1,70 ± 0,07
	4,5	1,69 ± 0,05
	5,0	1,46 ± 0,05
	5,5	1,52 ± 0,08
	6,0	1,30 ± 0,07

* It has been verified that the reaction $D + n \rightarrow 2n + p$ (*14.15*) has a negligible cross-section.

scatterer, have not been corrected for the effect of binding. The reduction factor necessary to get the free deuteron cross-section is hard to estimate; it probably lies between 0,5 and 0,7 (the lower limit (*6.213–29b*) being 0,44).

The *angular distribution* of the recoil deuterons * has been studied in the case of scattering of $D(d,n)$-neutrons. KRUGER *et al.* [38], using the cloud chamber method (KRUGER *et al.* [37]; *6.42*) to investigate the angular range $\Theta' = 0° \ldots 80°$ for an energy of 2,6 MeV of the incident neutrons, found a peak in the angular dependence of the differential cross-section at about 60°; but this result, based on the analysis of only 328 recoil deuterons does not seem to agree with BARSCHALL and KANNER's [40] ionization chamber investigation with neutrons of energy 2,44 ... 2,5 MeV, which do not show any departure from isotropy in the barycentric system in the range $\Theta' = 30° \ldots 55°$, i.e. $\vartheta = 70° \ldots 120°$. Recently, this investigation has been extended by COON, DAVIS and BARSCHALL [46*a*, *b*] to the whole range $\Theta' = 0° \ldots 55°$, i.e. $\vartheta = 70° \ldots 180°$. They found that the differential cross-section, as a function of ϑ, while remaining constant for $\vartheta = 70° \ldots 110°$, rises from $\vartheta = 110°$ to a peak value for $\vartheta = 180°$, about twice as large as the value for $\vartheta = 110°$. Finally, DARBY and SWAN [48], analyzing 902 recoil tracks by Dee and Gilbert's method (*6.42*), find, for neutrons of 2,53 MeV energy, an angular distribution qualitatively similar to that just described, but with a relatively much higher peak at $\vartheta = 180°$.

From the theoretical point of view, the whole problem has been thoroughly discussed by BUCKINGHAM and MASSEY [41] **. Their work is based on the general equation (7) of the resonating group method; for the distance dependence of the nuclear potential, they adopt the exponential type, with the constants of table *6.432*. The deuteron eigenfunction is then given by *5.211*. The wave-amplitude $F(\vec{x})$ is expanded in harmonics:

$$F(\vec{x}) = \sum_l \frac{1}{r} f_l(r)\, P_l(\cos\vartheta); \tag{20}$$

* The relation (*6.212–20*) between the angle of recoil Θ' in the laboratory system and the scattering angle ϑ in the barycentric system is independent of the masses of the collision partners. The scattering angle Θ in the system in which the scatterer of mass AM is at rest is related to ϑ by the general formula

$$\operatorname{tg}\Theta = \sin\vartheta \Big/ \left(\cos\vartheta + \frac{1}{A}\right)$$

whence

$$d\Omega_{\text{bar}} = \left[\frac{2}{A}\cos\Theta + \frac{A^2 + \cos 2\Theta}{A\sqrt{A^2 - \sin^2\Theta}}\right] d\Omega_{\text{lab}}.$$

** See also MASSEY and BUCKINGHAM [47*a*].

similarly, the kernels P, Q, N are expressed by series of harmonics of the type

$$Q(\vec{x},\vec{x}') = \sum_l \frac{2l+1}{4\pi r r'}\, q_l(r,r')\, P_l(\cos\vartheta), \tag{21}$$

ϑ being the angle between the directions of $\vec{x}$ and $\vec{x}'$. The amplitudes $f_l(r)$ then satisfy integro-differential equations of the same general form as (7):

$$\begin{aligned} &\frac{d^2 f_l}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right] f_l = \\ &\frac{4M}{3\hbar^2}\left[\alpha U(r) + (\tfrac{4}{3})^3 \int \left(\beta q_l(r,r') + \gamma\left[p_l(r,r') + (\varepsilon' - \tfrac{5}{3}\varepsilon_0)\, n_l(r,r')\right]\right) f_l(r')\, dr'\right], \end{aligned} \tag{22}$$

$$k^2 - \frac{4M}{3\hbar^2}\varepsilon';$$

for positive ε', they have the asymptotic form

$$f_l \sim \sin\left(kr - \tfrac{1}{2} l\pi + \delta_l\right). \tag{23}$$

The phases δ_l, for both doublet and quartet waves, have to be determined by numerical integration of (22), the kernels q_l, p_l, n_l being themselves evaluated by numerical methods. This formidable task has been carried out by the authors for four values of the kinetic energy of the impinging

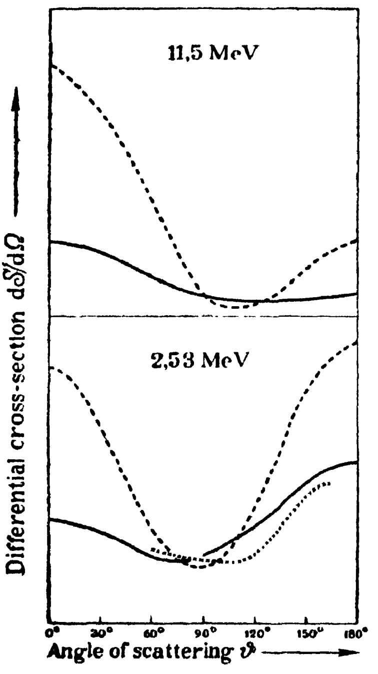

Fig. 14.12–1. Differential scattering cross-section of deuterons for neutrons. The differential cross-section (in arbitrary units) is given in terms of the scattering angle ϑ for neutrons of 11,5 MeV (upper curves) and 2,53 MeV (lower curves) kinetic energy; the full curves refer to the symmetrical theory, the dotted ones to the neutral theory. The crossed curve represents the observations of COON and BARSCHALL [46b].

neutrons, covering the range 0,25 ... 11,5 MeV (in the laboratory system), and for three different choices of the spin and charge dependence of the nuclear potential. For the energies involved, only S- and P-waves ($l = 0, 1$)

need be considered. The three forms of nuclear interaction investigated include the neutral and symmetrical theories (*8.32*–16, 17), to which we shall confine ourselves here *. The information thus provided is indeed very valuable, since the angular distributions predicted by these two theories turn out to be widely different, even for relatively low neutron energies (fig. *14.12*–1). Over the wide energy range surveyed by the calculations, the *P*-phases turn out to be much smaller on symmetrical than on neutral theory, with the result that the angular distribution is more uniform in the former than in the latter case.

The empirical material can be compared — disregarding the slight difference between the respective neutron energies — with the lower curves of fig. *14.12*–1. Unfortunately, the discrepancy mentioned above between Coon and Barschall's results on the one hand and Darby and Swan's on the other is so large as to preclude any discrimination between the two theoretical possibilities. In fact, while the first named experiments, as shown on the figure, tend to support the predictions of the symmetrical

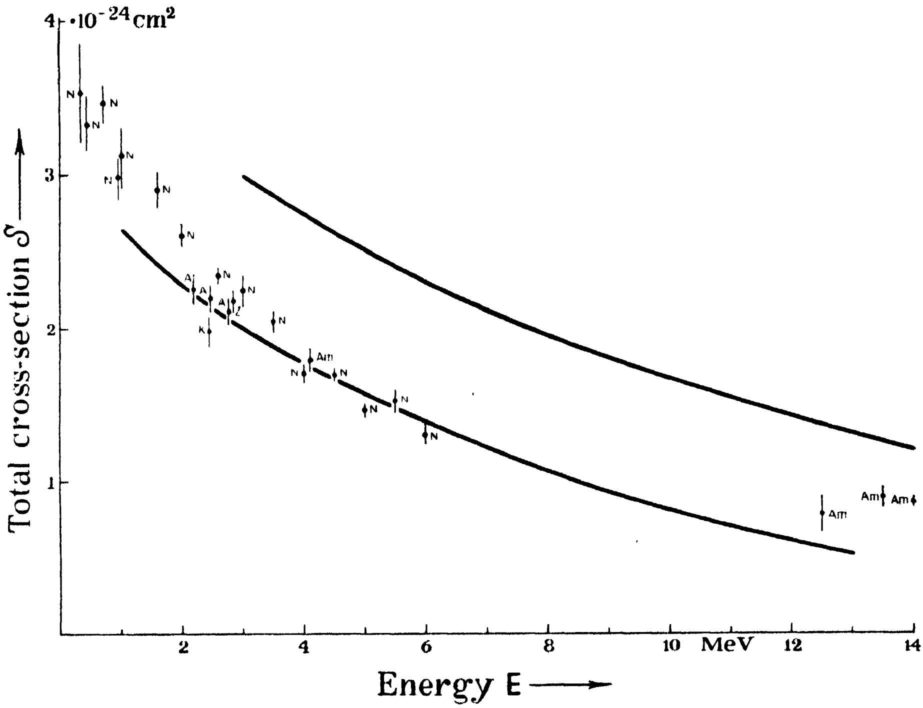

Fig. **14**.12–2. Total elastic scattering cross-section of deuteron for neutrons, in terms of the kinetic energy of the neutrons (in the laboratory system). Lower curve according to symmetrical theory, upper curve according to neutral theory. Experimental points:
A Aoki — *K* Kikuchi and Aoki — *Z* Zinn, Seely and Cohen.
Am Amaldi *et al.* — *N* Nuckolls, Bailey *et al.*

* The third type, a combination of Majorana and Heisenberg forces, yields results very similar to those of the symmetrical type.

theory, those of Darby and Swan seem to fit nicely the curve corresponding to the neutral theory.

A more definite conclusion can, however, be derived from the consideration of the total scattering cross-section. The theoretical values of this quantity as function of neutron energy, on both the neutral and the symmetrical theory, are illustrated by fig. *14.12–2*, in which the relevant experimental results listed in the above table have also been plotted. The general trend of these results undoubtedly points to the range of values expected on symmetrical theory. The analysis of the saturation properties of the nuclear interactions which, as we have seen (*11.33*), also leads us to this theory, cannot claim the simplicity and directness of the evidence just discussed. It thus appears that *the analysis of the scattering of neutrons by deuterons furnishes at present the weightiest argument in favour of the symmetrical form of nuclear interaction.* In view of the importance of this conclusion, it is highly desirable that the experiments be repeated and extended with increased accuracy.

Further calculations of the S-wave scattering cross-section for different values of the range of the nuclear potential show that at higher neutron energies this quantity does not depend very much on the value assumed for the range, but that for very slow neutrons (for which it represents practically the whole cross-section) it becomes quite sensitive to this value: it increases with increasing range. Unfortunately, the slow neutron cross-section is, as shown by the table above, not sufficiently well known experimentally to allow of a determination of the range; at any rate, the range value derived from the proton-proton scattering data would seem not to be inconsistent with the neutron-deuteron evidence.

An independent calculation of the S-wave scattering cross-section for the neutron energy range 0 ... 3 MeV has been performed by HÖCKER [42], using essentially the same method, but somewhat different assumptions on the nuclear potential. He uses the symmetrical form of interaction, adopting a Gauss potential with quite inadequate constants, viz. $J = 26{,}4$ MeV, $\varkappa^{-1} = d$, $q = 0{,}5$; he solves the integro-differential equation (22) for f_0 by a variational method. The values he gets for the cross-section are rather too low.

14.13. *Scattering of protons by deuterons.* The scattering of protons by deuterons is in serious need both of experimental investigation and of theoretical treatment. The only available measurements — discarding older ones of TUVE *et al.* [36] — are those of TASCHEK *et al.* [42, 47], with protons in the energy intervals 0,2 ... 0,3 MeV and 0,825 ... 3,53 MeV (using the same apparatus as in the study of proton-proton scattering, *7.12*), and of POWELL *et al.* [47*b*], who investigated by the photographic method (*6.42*) the scattering of 4,2 MeV protons in deuterium and of 6,7 MeV deuterons in hydrogen. A detailed phase-shift analysis of the main data has been given by CRITCHFIELD [48]. The theory of the process

is dealt with rather summarily in two older papers by PRIMAKOFF [37] and OCHIAI [37]; however, more up-to-date calculations by MASSEY and BUCKINGHAM [48], along the same lines as their investigation of the neutron-deuteron system, are in progress. In the preliminary note quoted, the authors compare their theoretical results with the ratio of observed to Rutherford scattering at various angles, interpolated for 1,85 MeV protons: at this energy, the symmetrical theory is found to lead to a remarkably good agreement, whereas the predictions from the neutral theory deviate markedly from the observed values.

14.14. *Radiative capture of thermal neutrons by deuterons.* The cross-section for the radiative capture of thermal neutrons by deuterons has been estimated by BORST and HARKINS [40] at $2 \dots 3 \cdot 10^{-28}$ cm^2 from measurements of the β-activity of the tritium (^{3}H) formed in this reaction. The effect is connected, just as the analogous capture of neutrons by protons (*6.52*), with a change in magnetic moment of the system. Its theory, worked out by HÖCKER [42] (on the assumptions mentioned at the end of *14.11*) and more exactly by BURHOP and MASSEY [48] (using the resonating group method outlined above), yields the right order of magnitude for the cross-section.

14.15. *Disintegration of the deuteron by proton or neutron impact.* Sufficiently fast nucleons can disintegrate a deuteron according to either of the reactions

$$D+p \to 2p+n \qquad (p)$$
$$D+n \to 2n+p \qquad (n).$$

Since a minimum energy of 2,18 MeV must be available in the barycentric system to make any one of these reactions possible, the threshold energy of the impinging nucleon is $2{,}18 \cdot \frac{3}{2} = 3{,}27$ MeV. The (p) reaction has been produced by BARKAS and WHITE [39*b*] with protons of 5,1 MeV. The deuterium gas was contained in a chamber with insulated lead lining for collecting the protons. The neutrons were slowed down by paraffin and measured with a silver detector. The cross-section was estimated at $\approx 1{,}4 \cdot 10^{-26}$ cm^2. The (n) reaction has hitherto not been studied directly *; in the course of their proton-neutron scattering experiments, AMALDI *et al.* [43] derived an estimate of the (n) cross-section for 14 MeV neutrons, viz. $4 \dots 9 \cdot 10^{-26}$ cm^2.

For the theoretical discussion of the (n) reaction, the reader is again referred to HÖCKER's paper [42]. The cross-section of the (p) reaction can, as also shown by Höcker, be derived from that of the (n) reaction by multiplying the latter by the penetrability of the deuteron barrier for

* The cloud chamber tracks of recoil particles from deuterium exposed to Rn–Be neutrons, observed by BAGGE [42], are of doubtful interpretation (see HÖCKER [42], AMALDI *et al.* [43]).

the proton (*2.1*). The theoretical results come out an order of magnitude larger than the crude empirical estimates just quoted.

14.2. Binding energies of the lightest nuclei

14.20. Numerous attempts have been made, starting from various assumptions on the form of the central potential of interaction between a pair of nucleons, to compute the binding energies of the nuclei ^{3}H, ^{3}He and ^{4}He. The discussion of *14.111*, leading to formula (19), shows that the energy of the ground state of ^{3}H depends primarily on the interaction constants $\varkappa$, J and q, i.e. on the range and the strengths of the effective potentials for even states. One can therefore try, in the first place, to adjust these parameters so as to account simultaneously for the observed values of the binding energy of ^{3}H and of the energies of the ^3S and ^1S levels of the deuteron (which we shall here call, for brevity, the "deuteron data"). It then remains to be seen whether agreement can also be obtained, in conformity with the charge-independence property, with the data from proton-proton scattering (the "p–p data") and with the observed mass-defect of ^{4}He. Of course, other groupings of the items of empirical evidence just enumerated are possible, e.g. adjusting the parameters to the deuteron and p–p data and testing the resulting potential by the calculation of the binding energies of ^{3}H and ^{4}He. In view of the margin left by the inaccuracy of the methods of computation of the latter quantities, these various possibilities may be helpful in elucidating different aspects of the consistency problem thus raised. In this section, we propose to review the discussions to which this problem has given rise and to draw our own conclusions. We begin with some preliminary remarks, carrying further the theory developed in *14.11* and *14.111*.

14.21. *Data and methods.* Let us first of all collect the data of interest. The binding energies of the ^2S ground state of ^{3}H and the ^1S ground state of ^{4}He, respectively, are

$$\begin{aligned} \varepsilon(^3\mathrm{H}) &= \varepsilon_3 = -8{,}3\ \mathrm{MeV} \\ \varepsilon(^4\mathrm{He}) &= \varepsilon_4 = -28{,}2\ \mathrm{MeV}. \end{aligned} \tag{1}$$

The triton ^{3}H is β-active, going over into ^{3}He with the emission of a rather slow electron: the maximum kinetic energy K_β of the electron spectrum is less than 15 keV (WATTS and WILLIAMS [46]). From the energy balance (*1.13*–1) of the process, with (*1.22*–5), we thus get

$$\varepsilon(^3\mathrm{He}) - \varepsilon(^3\mathrm{H}) = 1{,}47\,m - K_\beta \qquad (K_\beta < 0{,}03\,m). \tag{2}$$

The calculation of this energy difference, by (*14.111*–18), affords a useful test of the accuracy of the method of approximation applied to the determination of ε_3. We may state at this point that in all the papers which

will be discussed below, this test has been carried out with a satisfactory outcome; this will dispense us from mentioning the treatment of ^{3}He in the sequel. Finally, we recall for later reference, that the ground state of ^{5}He is a virtual level, involving the spontaneous decay of this nucleus into a neutron and an α-particle; its energy is 0,8 MeV above that of the latter, i.e.

$$\varepsilon(^5\mathrm{He}) = -27{,}6\ \mathrm{MeV}. \tag{3}$$

The methods of computation at our disposal are the resonating group method (*14.11*), the variational method (*14.111*), the variation-iteration method (*5.24*) and the ordinary perturbation method. As regards Svartholm's variation-iteration scheme, it has hitherto only been applied to the problem in hand, with very encouraging results, by Svartholm himself. The resonating group method, which might be expected to be the most reliable one, has been put to the test by Miss WAY [39*b*] in the case of ^{3}H and ^{5}He: the conclusion of this study is rather disappointing. In the case of ^{3}H, Miss Way carried through the whole calculation according to the general prescriptions of the method (*13.3*), but found that this yielded hardly better results than a simplified procedure, consisting in choosing for the wave-amplitude $F(\vec{x})$ a suitable analytic expression with variational parameters, thus reducing the problem to the usual variation principle *. She applied this method also to the $^2P(1s)^4\,2p$ and $^2S[(1s)^4\,2s + (1s)^3\,(2p)^2]$ configurations of ^{5}He. Both nuclei had been treated by the ordinary variational method, using very nearly the same specification of the nuclear potential (Gauss type), viz. ^{3}H by MARGENAU and WARREN [37] and ^{5}He by TYRRELL [39]. This refined variational treatment, involving trial functions of complicated form, proved superior to the resonating group method. In particular, only the former treatment is able to bring out the existence of a ^2S level of ^{5}He with nearly the same energy as the ^2P state (which is probably lower). The failure of the resonating group method in the case of ^{3}H might be attributed to the neglect of the polarization of the deuteron cluster by the loose neutron; but in the case of ^{5}He, such polarization effects should be unimportant.

The methods most extensively used can be described as more or less elaborate applications of the variation principle; sometimes the perturbation method has also been employed. A critical study of the technical aspects

* In the course of his treatment (by the resonating group method) of the radiative capture of neutrons by deuterons (*14.14*), Höcker likewise tries to set up a simple expression for the amplitude $F(\vec{x})$ corresponding to the ground state of the triton (final state of the capture process). He does not, however, determine the "best" parameter values by the variation principle, but contents himself with fixing them in such a way that the expectation value of the energy is equal to the observed value. Burhop and Massey, on the other hand, replace the kernel of the integro-differential equation for $F(\vec{x})$ by an approximate expression, making possible an analytic solution of the equation; this procedure yields (fortuitously) an excellent result.

of the calculations and convergence questions involved will be found in a paper by WARREN and MARGENAU [37]. We shall here only point out that a rapid, but not always very reliable, orientation may be gained with the help of a peculiar mode of estimate of the energies ε_3, ε_4 of ^{3}H and ^{4}He, known as the reduction to "equivalent two-body problems", the idea of which goes back to WIGNER [33a]. The simple form in which it will be presented here is due to RARITA and PRESENT [37]. Let us first write down the variation principle, pertaining to ^{4}He, in a simplified form analogous to equation (*14.111*–19) for ^{3}H. We adopt a wave-function entirely symmetrical in the space coordinates:

$$\Psi = \varphi(1234)\frac{1}{\sqrt{4!}}\sum_P(\pm)\,P\,[{}^3(\tau)_1(\overline{12})\cdot{}^3(\tau)_{-1}(\overline{34})\cdot{}^1(\sigma)_0(\widetilde{12})\cdot{}^1(\sigma)_0(\widetilde{34})]; \quad (4)$$

in this formula, the summation extends over all permutations P of the numbers 1234, affected with the + or — sign according as they are even or odd. One then easily finds:

$$\varepsilon = \frac{1}{N}\left[-4\frac{\hbar^2}{2M}K' - 6\,\frac{1+q}{2}\,J\right]. \quad (5)$$

the symbols N, K', J being defined in complete analogy to those referring to ^{3}H: in the formulae (*14.111*–17*a*, 17*b*, 19), one has just to replace φ (123) by φ(1234) and to integrate also over $dv^{(4)}$. To this ε must be added the average Coulomb energy C/N, with C analogous to the integral in (*14.111*–18). Now, it turns out that if the trial functions φ (123) or φ(1234) are taken as products of "Gaussian" functions of the type $\exp(-\frac{1}{2}\mu r^{(ik)2})$:

$$\varphi = \exp\left(-\tfrac{1}{2}\mu\sum_{i>k} r^{(ik)2}\right), \quad (6)$$

both equations (*14.111*–19) and (5) reduce to the same simple form, which is just the expression of the variation principle for a two-body problem. Elementary integrations yield in fact the following results:

$$\begin{aligned}
&{}^2\mathrm{H}: && \varepsilon = \tfrac{3}{2}\mu - J\,W(\mu)\\
&{}^3\mathrm{H}: && \varepsilon = \tfrac{9}{2}\mu - 3\,\frac{1+q}{2}\,J\,W(\tfrac{3}{2}\mu)\\
&{}^4\mathrm{He}: && \varepsilon = 9\mu - 6\,\frac{1+q}{2}\,J\,W(2\mu),
\end{aligned} \quad (7)$$

with

$$W(\mu) = \frac{4}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,w\!\left(\frac{\varkappa x}{\sqrt{\mu}}\right)x^2\,dx. \quad (8)$$

In other words, the variation problems for ^{3}H and ^{4}He are identical with

that of a two-nucleon system; one has just to establish between the respective potential constants the following correspondence:

$$\begin{array}{llll} {}^2\text{H}: & \varkappa\,, & J & \\ {}^3\text{H}: & \varkappa\sqrt{2}\,, & 3\dfrac{1+q}{2}J & \qquad (9) \\ {}^4\text{He}: & \varkappa\sqrt{3}\,, & 6\dfrac{1+q}{2}J. & \end{array}$$

The method of the "equivalent two-body problem" consists in taking as an estimate of the energy ε_3 or ε_4 the *exact* eigenvalue of the two-body wave-equation involving the corresponding interaction parameters as given by (9). This procedure has the advantage of simplicity, but it clearly lacks any theoretical justification. For some forms of distance dependence of the nuclear potential (Gauss, exponential), it may yield rather close approximations; but in other instances, e.g. the meson potential, in which the Gaussian function is but a poor representation of the deuteron eigenfunction, it is far less reliable.

14.22. *Discussion of results.* Computations of varying accuracy have been carried out for different types of nuclear interaction, viz. the exponential potential (RARITA and PRESENT [37]), the Gauss potential (MARGENAU and collaborators [37, 38*a*], SVARTHOLM [45]) and the meson potential (HYLLERAAS and RISBERG [41], BROWN [39], SVARTHOLM [45], FRÖHLICH *et al.* [46, 47], HUMBLET). Insofar as the simplified variation problems (*14.111*–19) and (5) are considered, or the calculation by some other method is limited to the first approximation, only the effective potentials for even states are concerned and the precise choice of the spin and charge dependence of the potential is immaterial (except for the charge independence property). For more accurate calculations, as well as for the investigation of excited states *, it is, strictly speaking, necessary to adopt a definite form: in the papers just quoted, either the symmetrical theory or a mixture of Majorana and Heisenberg forces has been used **; however, in practice it hardly makes any difference which choice is made. In fact, when judging the degree of accuracy of such calculations, one should not forget that they imply the complete neglect of possible non-central forces and, above all, of the contributions from direct 3-body and 4-body interactions. Especially the latter circumstance has not been properly recognized until recently; otherwise, the laborious investigations of which an account will now be given would probably have been led into more promising channels.

The consistency problem was raised for the first time in connexion

* MARGENAU and TYRRELL [38*a*] discuss some excited states of the four-nucleon (^{4}He) system.

** Svartholm uses a mixture of Wigner and Bartlett forces and applies a correction to go over to the Majorana-Heisenberg case.

with Rarita and Present's treatment of the *exponential type of potential.* They found a set of potential constants (those given in table *6.432*) consistent with both ε_3 and deuteron data; we may add (*7.13*) that they turn out to be consistent also with the *p–p* data. But when the ^{4}He energy $|\varepsilon_4|$ is computed with these constants, it comes out considerably *too large.*

Margenau and his collaborators then took up the problem using a *Gauss potential.* By very elaborate variational calculations, they determined a set of parameters:

$$\varkappa^{-1} = 2{,}25 \cdot 10^{-13}\,\text{cm}, \qquad J = 35{,}6\ \text{MeV}, \qquad q = 0{,}6, \tag{10}$$

with which they were able to account for ε_3 and the deuteron data. The binding energy $|\varepsilon_4|$, however, came out slightly too small; the observed value of ε_4 could be obtained by increasing the parameter q to 0,65, but with such a high (absolute) value of the ^1S effective potential, the ^1S level of the deuteron becomes an actual state of binding. Moreover, when the *p–p* data became available, they proved clearly incompatible with the constants (10): a smaller range was required (*7.13*). Resuming the calculation, by his variation-iteration method, with this smaller range, Svartholm could further show that the adjustment of the potential to the deuteron data and ε_3 was possible, but again yielded too large a value of $|\varepsilon_4|$. Summing up, the two following conclusions emerge:

(1) a set of parameters compatible with the deuteron and *p–p* data and with ε_3 yields too large a binding of ^{4}He;

(2) a set of parameters yielding the right binding energies of ^{2}H, ^{3}H and ^{4}He involves a larger range than allowed by the *p–p* data and gives rise to a bound ^1S state of the deuteron.

Both these conclusions have been confirmed in the case of the *meson potential.* As shown by Hylleraas and Risberg, the second one readily follows from the application to this case of the "equivalent two-body" mode of estimate; the numerical results obtained by these authors are probably exaggerated by the inadequacy of the method, but their relations, as formulated in the above statement (2), are undoubtedly correct. The first conclusion results from the work of Brown and Svartholm: the former finds that the meson potential corresponding to the mass-value $M_m = 211\ m$, — the strengths $^3J^{(0)}$, $^1J^{(0)}$ being given in terms of M_m by (*6.432*–8), — will suit both ε_3 and deuteron data; Svartholm then shows that an excessive binding of ^{4}He results. Moreover, Svartholm establishes the same conclusion for the mass-value $M_m = 326\,m$ suggested as a possible one by the *p–p* data (*7.13*).

The first paper by Fröhlich *et al.* [46] contains a determination of the meson mass from the deuteron data and ε_3, very similar to that of Brown, though based on the simplified variation principle (*14.111*–19); this leads to $M_m = 225\,m$. An improved calculation, carried out in the second paper [47], yields $M_m = 210\,m$ in complete agreement with Brown *. On the

* Fröhlich and his collaborators adopt the symmetrical theory, while Brown assumes a mixture of Majorana and Heisenberg forces.

other hand, it is very strange that — in contrast with all previous results, and especially with Svartholm's — the value of $|\varepsilon_4|$ computed by Fröhlich *et al.* is smaller than the observed one (less than half the experimental value). There is presumably some error in this computation, which, owing to the special form adopted for the trial wave-functions, is very intricate. At any rate, a renewed calculation of $|\varepsilon_4|$ by HUMBLET, using the usual "Gaussian" trial functions, yields results much nearer to Svartholm's. By varying the mass of the meson, Humblet shows that the value of the ^{4}He binding energy is very sensitive to this parameter, and increases (in absolute value) with increasing meson mass; it is true that he finds an $|\varepsilon_4|$ *smaller* than the observed value for M_m as high as $\approx 240\ m$, which slightly disagrees with Svartholm's calculation, but on the whole his results do not deviate from the general trend expressed by the above statements (1) and (2).

Now, what is the significance of these statements? Considering (2), it is clear that we must renounce any consistent account of the binding energies of the lightest nuclei by means of central two-body interactions alone. Conclusion (1) then embodies the most that we can expect in the way of consistent interpretation on this basis and tells us at the same time in which direction a first discrepancy arises. This discrepancy has often been commented on with undue pessimism, when one reflects on the narrowness of the fundamental assumption just recalled about the nature of the nuclear forces at play. It is indeed reasonable to suppose that the excess of binding of ^{4}He should be removed by forces which make their first appearance in a system of four nucleons, but are inoperative in systems composed of fewer particles; i.e. by direct four-body interactions, which in the special case of ^{4}He should have the effect of a repulsion. This inference, drawn both by Svartholm and by Fröhlich *et al.* *, seems very cogent indeed. Of course, it still needs quantitative verification; but in its present state, meson field theory is not yet ripe for such problems.

14.23. *Argument for like-nucleon forces.* Early calculations of binding energies of ^{3}H and ^{4}He (B & B, §§ 20—21) were carried out with the help of a nuclear potential not presupposing the charge independence property, and the strengths of neutron-proton and like-nucleon potentials were determined independently so as to fit deuteron data as well as ε_3 and ε_4. In this way, the existence of forces between like nucleons received a further confirmation and their order of magnitude was found to agree with that derived from the p–p data.

14.24. *Stationary states of ^{6}Li and ^{6}He.* According to rough deductions from the quasi-atomic model (*11.41*), the ground state of the stable odd nucleus ^{6}Li should be a ^3S state, and the first excited level a ^3D state

* Owing to their aberrant result on ε_4, these authors are naturally led to postulate a 4-nucleon *attraction*. On the other hand, they rightly observe that since ε_4 appears as the difference of two large quantities, viz. the kinetic and potential energies, nearly balancing each other, a relatively small contribution from 4-body forces to the potential energy would be sufficient to produce the big change required in their ε_4-value.

(*10.22*). Detailed calculations by MARGENAU and CARROLL [38*b*], combining variational and perturbation methods, confirmed this, but disclosed at the same time the unreliability, from a quantitative point of view, of first order results concerning the spacing of levels of different orbital quantum numbers: in fact, the second order perturbations decrease considerably the ^3D–^3S interval, without, however, reversing the order of the levels. A similar effect has been found by Tyrrell in his calculation (*14.21*) of the ^2S–^2P interval in ^{5}He.

The energy difference between the ground states of the isobaric nuclei ^{6}He and ^{6}Li, connected by a β-transition $^6\mathrm{He} \xrightarrow{\beta} {}^6\mathrm{Li}$, can be deduced from the limiting energy of the β-rays; after allowing for the Coulomb energy, there remains a difference in proper nuclear energy amounting to $3{,}8 \pm 0{,}6$ MeV. This arises, of course, from the fact that the ^{6}He configuration, due to the exclusion principle, is a ^1S one. Assuming charge independence of the nuclear forces, we may say that the difference in question represents the ^1S–^3S separation for 6-nucleon systems. Its theoretical calculation has been carried out by MARGENAU [39]; in this case, in which we are dealing with states of the same orbital momentum, the second order effects have actually little influence. In principle, we might therefore hope to derive from comparison with experiment definite information about the spin dependent part of the nuclear potential. However, — quite apart from the uncertainty regarding possible contributions from other than central interactions, — the empirical value itself is too inaccurate to allow of any such deductions.

SUMMARY OF PART III

In this Part we have been concerned with two main lines of argument. On the one hand, we have discussed how far it is possible to account for the properties of complex nuclei, according to various *structural models*, on the assumption of a central interaction between pairs of nucleons. In particular, a tendency has been revealed towards the formation, within such a nucleus, of transient clusters of nucleons, especially α-clusters (*12.23*); and nuclear models have been described in which account is taken of this *cluster structure* (*13.3*).

On the other hand, we have considered the additional evidence from complex nuclei concerning the exact form of the *isotopic factor of the law of nuclear interaction*. In this respect, the saturation properties of nucleon binding (*11.33*) and the scattering of neutrons by deuterons (*14.12*) have both been found to yield evidence in favour of the *symmetrical* form of nuclear potential (*8.31*). This conclusion completes the determination of the elementary law of force, insofar as it may be — at least in first approximation — regarded as a central two-body interaction.

There are, however, quite definite indications as to the necessary corrections to be applied to this description. The computation of the binding energies of the lightest nuclei has revealed the far from negligible role played by many-body forces (*14.22*). Moreover, as we know (*6.12*), a certain amount of non-central coupling is necessary to explain the electromagnetic properties of the deuteron. While little can be said at present about many-body forces, considerable work has already been devoted to a closer study of non-central interactions. Our last task will now be to survey the preliminary results, as yet inconclusive, pertaining to this important question.

PART IV

NON-CENTRAL AND NON-STATIC COUPLINGS

CHAPTER XV

GENERAL TYPES OF NON-CENTRAL AND NON-STATIC NUCLEAR FORCES

15.1. General empirical evidence

The analysis of the electromagnetic properties of the ground state of the deuteron (*6.12, 6.121, 6.122*) has made it clear that the proton-neutron interaction involves a non-central term, as the result of which the ^3S ground state contains a small admixture of $^3D_1$ state. The occurrence of the non-central coupling is a direct consequence of the existence of an electric quadrupole moment in this state. That the admixture of ^3D state in the case of the deuteron is very small can be concluded from a comparison of the actual magnetic moment μ_d with the value $\mu_d^0 = \mu_n + \mu_p$ pertaining to a pure ^3S state. The values of the magnetic moments of the other stable odd nuclei confirm the inferences drawn from the deuteron data and illustrate the increasing importance of the non-central couplings with increasing complexity of the nucleus (*A2.26*).

Other evidence from nuclei more complex than the deuteron, such as that derived from the fine structure of the levels or, more indirectly, from the magnetic moments, is, as we shall see, much less definite. In fact, the general conclusions which the preceding study of heavier nuclei from the standpoint of central interaction permits do not open very encouraging prospects of a more detailed analysis of this interaction: on the one hand, the gross features of nuclear properties are to a surprising extent insensitive to even large variations of the law of nuclear force; on the other, the investigation of finer details, which might in principle yield useful information on the precise form of this law, would require an accuracy of theoretical treatment which present methods would generally not permit. However, very general and elementary arguments of covariance will at least carry us a first step forward, by enabling us to enumerate the simplest forms of interaction potentials possible *a priori*.

15.2. Survey of nuclear interactions

15.21. *Static and first order interactions.* In trying to determine the various possible types of nuclear forces, we shall first of all neglect many-body forces (*1.34*) and restrict the discussion to interactions between pairs of nucleons. The question is, of course, essentially non-relativistic: a relativistic formulation of interaction laws is not possible without the help of the field concept. The introduction of an interaction potential is only justified insofar as the velocities of the constituent nucleons remain on the average sufficiently small (*2.3*), and we may therefore consistently limit

ourselves to expressions involving no higher power of the momenta than the first. We need not consider any explicit dependence on the variables ϱ_i, since the latter variables can always be reduced to the momenta. We have thus to construct interaction operators of the general form $\frac{1}{2}\Sigma\mathcal{V}^{(ik)}$, in which $\mathcal{V}^{(12)} = \mathcal{V}(Q^{(1)}, Q^{(2)})$ is a linear function of the momenta of the two nucleons (1) and (2), symmetric with respect to the sets of coordinates $Q^{(1)}$, $Q^{(2)}$, and further satisfying all invariance requirements listed under (*a*) ... (*f*) in *4.31*. The following discussion has been given in two papers by WIGNER [37*a*] and EISENBUD and WIGNER [41].

Let us first fix our attention on the rotation and spatial symmetry covariance. As regards the spatial variables, we have at our disposal (suppressing the indices $^{(12)}$)

$$
\begin{array}{lll}
\text{2 scalars:} & r = |\vec{x}|, & \vec{x}\cdot\vec{p}; \\
\text{2 polar vectors:} & \vec{x}\ , & \vec{p}\ ; \\
\text{1 axial vector:} & \vec{x}\wedge\vec{p}; & \\
\text{2 tensors:} & T^{(x)}_{lm} = x_l x_m, & T^{(xp)}_{lm} = x_l p_m.
\end{array}
\tag{1}
$$

With the axial vectors $\vec{\sigma}^{(1)}$, $\vec{\sigma}^{(2)}$ we can further form

$$
\begin{array}{ll}
\text{2 scalars:} & 1,\ \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}; \\
\text{3 axial vectors:} & \vec{\sigma}^{(1)}+\vec{\sigma}^{(2)},\quad \vec{\sigma}^{(1)}-\vec{\sigma}^{(2)},\quad \vec{\sigma}^{(1)}\wedge\vec{\sigma}^{(2)}; \\
\text{1 tensor:} & T^{(\sigma)}_{lm} = \sum_n \sigma^{(1)}_{ln}\sigma^{(2)}_{nm}\,.
\end{array}
\tag{2}
$$

In writing the tensor components, we denote by x_i, p_i ($i = 2, 2, 3$) the components of $\vec{x}$ and $\vec{p}$ and likewise we put

$$\sigma_{12} = \sigma_z \qquad \text{(cycl)}.$$

We have to combine the various quantities (1) and (2) in such a way as to fulfil in the first place the requirements (*b*), (*c*), (*d*) of dynamical and spatial invariance, which do not involve the isotopic variables. To obtain invariants from the tensors $T^{(x)}$, $T^{(xp)}$, $T^{(\sigma)}$ in the most convenient form, it is best to start from the traceless tensors $T_{lm} - \frac{1}{3}\delta_{lm}T$ (T denoting the trace $\sum_l T_{ll}$). If $T^{(\alpha)}$, $T^{(\beta)}$ are any two of our tensors, $T^{(\sigma)}$ being symmetrized if necessary, we define an invariant by

$$
\begin{aligned}
I^{(\alpha,\beta)} &= \sum_{l,m}(T^{(\alpha)}_{lm} - \tfrac{1}{3}\delta_{lm}T^{(\alpha)})(T^{(\beta)}_{lm} - \tfrac{1}{3}\delta_{lm}T^{(\beta)}) \\
&= \sum_{l,m} T^{(\alpha)}_{lm}T^{(\beta)}_{lm} - \tfrac{1}{3}T^{(\alpha)}T^{(\beta)};
\end{aligned}
\tag{3}
$$

the traces of the tensors we have to use are

$$T^{(x)} = r^2, \quad T^{(xp)} = \vec{x}\vec{p}, \quad T^{(\sigma)} = -2\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}. \tag{4}$$

An easy discussion leads to the following independent types of invariants:

$$1, \quad \vec{\sigma}^{(1)}\vec{\sigma}^{(2)} \tag{5a}$$

$$r^2\,\mathfrak{D}^{(12)} = (\vec{x}\vec{\sigma}^{(1)})(\vec{x}\vec{\sigma}^{(2)}) - \tfrac{1}{3}r^2\vec{\sigma}^{(1)}\vec{\sigma}^{(2)} \tag{5b}$$

$$(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot(\vec{x}\wedge\vec{p}) \tag{5c}$$

$$(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})\cdot(\vec{x}\wedge\vec{p}) \tag{5d}$$

$$(\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)})\cdot(\vec{x}\wedge\vec{p}); \tag{5e}$$

$$\begin{gathered}\vec{x}\vec{p}, \qquad (\vec{x}\vec{p})(\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}), \qquad \vec{x}\vec{p}\,\mathfrak{D}^{(12)}, \\ \tfrac{1}{2}[(\vec{x}\vec{\sigma}^{(1)})(\vec{p}\vec{\sigma}^{(2)}) + (\vec{x}\vec{\sigma}^{(2)})(\vec{p}\vec{\sigma}^{(1)})] - \tfrac{1}{3}(\vec{x}\vec{p})(\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}).\end{gathered} \tag{6}$$

Each of these may be multiplied by an arbitrary function of the distance r.

Turning to the charge invariance requirement (f), we see that it can be satisfied by six distinct operators built up with the help of the isotopic variables $\boldsymbol{\tau}^{(1)}$, $\boldsymbol{\tau}^{(2)}$, viz.

$$1, \quad \boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}, \quad \tau_3^{(1)}\tau_3^{(2)}, \quad \tau_3^{(1)} + \tau_3^{(2)} \tag{7a}$$

$$\tau_3^{(1)} - \tau_3^{(2)} \tag{7b}$$

$$(\boldsymbol{\tau}^{(1)} \wedge \boldsymbol{\tau}^{(2)})_3 = 2i\,(\Pi^{(1)\dagger}\,\Pi^{(2)} - \Pi^{(1)}\,\Pi^{(2)\dagger}) = 2\,\Pi^{(12)}. \tag{7c}$$

In $(7c)$ use has been made of the notation introduced in (*4.41–2*). The most general interaction operator for a pair of nucleons will be a linear combination of these six operators, the coefficients of which will be suitable functions of the other variables. The allowed combinations of isotopic factors (7) with spatial invariants (5) and (6) are essentially limited by the requirement (e) of invariance for a change of sign of the time. On that account, the isotopic factors $(7a)$ and $(7b)$ can only be combined with the invariants of types $(5a)$ to $(5d)$; but since the resulting expression must also be invariant for an interchange of *all* coordinates of the two nucleons, $(7a)$ is compatible only with $(5a)$, $(5b)$, $(5c)$, while $(7b)$ belongs to $(5d)$ only. Since $(7c)$ changes sign when the Π's and $\Pi^\dagger$'s are interchanged, it can only be associated with either $(5e)$ or (6), which likewise change sign when the signs of the $\vec{p}$'s and $\vec{\sigma}$'s are changed; but the symmetry with respect to the sets of coordinates of both nucleons excludes all invariants of types (6). The requirement (a) of being Hermitian, which is the last we still have to consider, is obviously fulfilled by all remaining combinations.

As a result of the preceding discussion, we are thus left with a relatively small number of possible forms of nuclear potentials, each of which can be represented by a product of three factors: one of them, expressing the distance dependence, remains arbitrary; the second one embodies the dependence on spin, angle and momentum variables, the third is the isotopic factor. The following table enumerates the various types, with the names and symbols by which they will henceforth be denoted; the notation $\vec{x}_0$ has been used for the unit vector $\vec{x}/r$.

15.21. *Types of static and first order nuclear interactions*

Dependence on spin, angle and momentum variables			Dependence on charge (isotopic factor)		
I.	central	$1, \quad \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$	any linear combination of the factors	1. neutral	1
II.	axial dipole	$\mathcal{D}^{(12)} = (\vec{\sigma}^{(1)}\vec{x}_0)(\vec{\sigma}^{(2)}\vec{x}_0) - \frac{1}{3}\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$		2. symm.	$\tau^{(1)}\tau^{(2)}$
III.	spin-orbit	$\mathcal{M}^{(12)} = (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})(\vec{x} \wedge \vec{p})$		3. charged	$T^{(12)}$
IV.	spin-orbit of the 2nd kind	$\mathcal{M}_2^{(12)} = (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})(\vec{x} \wedge \vec{p})$			$\tau_3^{(1)}\tau_3^{(2)}$
V.	spin-orbit of the 3rd kind	$\mathcal{M}_3^{(12)} = (\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)})(\vec{x} \wedge \vec{p})$			$\Pi^{(12)} = \frac{1}{2}(\tau^{(1)} \wedge \tau^{(2)})_3$

With respect to the velocity dependence, we distinguish static and non-static couplings; the static ones may be either central (type I) or non-central (type II), while the non-static ones are all angle dependent. The interaction $\mathcal{D}^{(12)}$ is the familiar expression occurring in the interaction energy of two magnetic dipoles (*6.13*–20): hence its name *axial dipole coupling*. The non-static interaction $\mathcal{M}^{(12)}$ also represents a type well-known especially in atomic theory under the name *spin-orbit coupling*. The names of the isotopic factors which can be associated with any of the types I, II, III, of interaction have been chosen with reference to meson field theory (*8.31*). The factor $\tau_3^{(1)} + \tau_3^{(2)}$ has been left out, since its occurrence is excluded by the equality of proton-proton and neutron-neutron nuclear interactions (*3.3*). The types IV and V of non-static couplings are only listed for the sake of completeness; they are indeed very unlikely to play any role owing to their peculiar isotopic factors, which have non-vanishing matrix-elements only for proton-neutron interaction and are thus incompatible with the charge independence property of the nuclear potential.

15.22. *Second order interactions.* The preceding survey must be completed by an examination of the interactions of second order in the nucleon velocities which are compatible with the requirement of Lorentz invariance of the Hamiltonian up to that order. This discussion will be of importance in showing, first, that a potential energy, function of the variables of the nucleon pair, may be introduced in the relativistic Hamiltonian of the two-nucleon system as long as effects of the first order only

are considered (*4.31*); secondly, that second order interactions are generally so small compared with those of lower order that it is permissible in most cases to neglect them. Attacked for the first time by INGLIS [36], the problem has later been treated more systematically by BREIT [37*a*, 38*a*], to whose papers the reader is referred for a detailed study. We shall here discuss in full a simple case by way of illustration and only briefly mention more general results obtained in essentially the same way.

Let us first consider the behaviour of the nucleon moving in a field of force described in unrelativistic approximation by a potential energy V_n: on the quasi-atomic model, this might be taken as the average interaction of the nucleon with the other constituent particles of a nucleus. At the same time we shall also introduce an electrostatic potential energy V_e, in order to bring out the characteristic differences in the treatments of the nuclear energy V_n and the electromagnetic energy V_e. The latter, in fact, behaves as the time-component of a four-vector; whereas, according to an elegant procedure suggested by FURRY [36], we may regard the nuclear energy term as a scalar, and accordingly add it to the mass term of the Hamiltonian. In this way, we get (*4.221–9*) the invariant expression

$$H = \varrho_1 \vec{\sigma}\vec{p} + T_- V_e + \varrho_3 (M + V_n) - \tfrac{1}{2}(g_s - 2T_-)\mu_0 \vec{\sigma}\vec{H}_e, \tag{8}$$

in which g_s denotes the Landé factor $2\mu_n$ or $2\mu_p$ of the nucleon (*A2.22–11*) and $\vec{H}_e$ the magnetic field acting on its anomalous magnetic moment. In fact, the energy contribution due to the anomalous moment, being invariant *, can be calculated in the system of reference in which the nucleon is momentarily at rest; in this system, the nucleon is acted upon by a magnetic field $\vec{H}_e = \vec{E} \wedge \vec{v}/c$, $\vec{E}$ being the electric field $-\frac{1}{e}$ grad V_e and $\vec{v} \approx c\vec{p}/M$ the velocity of the nucleon in the original system of reference.

The wave-equation derived from the Hamiltonian (8) can be reduced in the usual way to an equation for the "large" components only (*4.21, 4.33*). The latter corresponds to the following reduced Hamiltonian, exact to the second order in the nucleon velocities:

$$\begin{aligned} H_{red} = {} & \frac{p^2}{2M} + T_- V_e + V_n - \frac{1}{2M}\left[(\varepsilon - T_- V_e)^2 - V_n^2\right] \\ & + \frac{i\hbar}{(2M)^2}\,\mathrm{grad}\,(V_n - T_- V_e)\cdot\vec{p} - \frac{\hbar}{(2M)^2}\,\vec{\sigma}\,[\mathrm{grad}\,(V_n - T_- V_e)\wedge\vec{p}] \\ & + \frac{\hbar}{(2M)^2}(g_s - 2T_-)\,\vec{\sigma}\,[\mathrm{grad}\,V_e \wedge \vec{p}]; \end{aligned} \tag{9}$$

In this formula, ε denotes the unrelativistic energy of the system. The last two terms can be combined to give a second order interaction

$$V^{(2)} = -\frac{\hbar}{(2M)^2}\,\vec{\sigma}\,\{\mathrm{grad}\,[V_n - (g_s - T_-)V_e] \wedge \vec{p}\}, \tag{10}$$

which, in the case of central potentials V_n and V_e, takes the form of a spin-orbit coupling

$$V^{(2)} = -\frac{\hbar}{(2M)^2}\,\frac{1}{r}\,\frac{d}{dr}\,[V_n - (g_s - T_-)V_e]\,\vec{\sigma}(\vec{x} \wedge \vec{p}). \tag{11}$$

* Cf. W. PAULI, *Hdb. d. Physik*, Bd. XXIV/1, 1933, p. 233.

INGLIS [36] gives a more physical analysis of this *relativistic spin-orbit coupling*, by observing that it consists of the superposition of two distinct contributions. In the first place, the magnetic dipole due to the spin of the nucleon, moving in the electric field $\vec{E}$, can be regarded, in the reference system R in which the particle is momentarily at rest, as acted upon by the magnetic field $\vec{H}_e$ and accordingly performing, in this reference system, a Larmor precession of angular velocity (divided by c)

$$\vec{\omega}_L = -\frac{e}{2M}\, g_s \vec{H}_e = \frac{g_s}{2M^2}(\text{grad } \mathcal{V}_e \wedge \vec{p}). \tag{12}$$

When we transform to the original system of reference, we have to introduce, owing to the acceleration of the nucleon, the well-known Thomas precession *, affecting the whole reference system R. Its angular velocity is

$$\vec{\omega}_T = \frac{1}{2}\frac{d}{cdt}\left(\frac{\vec{v}}{c}\right) \wedge \frac{\vec{v}}{c} = -\frac{1}{2M}\text{grad}\,(\mathcal{V}_n + \mathrm{T}_-\mathcal{V}_e) \wedge \frac{\vec{p}}{M}. \tag{13}$$

The total (kinetic) energy change due to these precessions amounts to $\frac{1}{2}\hbar\vec{\sigma}(\vec{\omega}_L + \vec{\omega}_T)$, which, according to (12) and (13), is just (10). While the nuclear potential $\mathcal{V}_n$ naturally gives rise only to a Thomas precession, the electric potential contributes also a Larmor term; as a result, the effect of $\mathcal{V}_n$ is enhanced by that of $\mathcal{V}_e$ in the case of a proton, counteracted in the case of a neutron: this follows indeed, according to (10), from the fact that $\mathcal{V}_n$ and $\mathcal{V}_e$ are of opposite signs (the former attractive, the latter repulsive), while $g_s - \mathrm{T}_-$ is positive for a proton, negative for a neutron.

As already stated, the Hamiltonian (8) or (9) might be used to describe, on the quasi-atomic model, the stationary states of an odd-mass nucleus in which all the constituent nucleons but one are in their lowest configuration with saturated spins. Such stationary states will be doublets, and in the absence of any other non-central coupling, the splitting of these doublets into their components $J = L \pm \frac{1}{2}$ would be entirely effected by the spin-orbit coupling $\mathcal{V}^{(2)}$. This effect is most readily vizualized by a comparison with the well-known case of an atomic system with one optical electron. The atomic problem is formally included in our formulae, if we put $\mathcal{V}_n = 0$, $g_s = 2$, $\mathrm{T}_- = 1$; according to (10), we have here

$$\mathcal{V}^{(2)}_{\text{atom}} = \frac{\hbar}{(2m)^2}\,\vec{\sigma}\,(\text{grad } \mathcal{V}_{e,\text{atom}} \wedge \vec{p}). \tag{14}$$

Since $\mathcal{V}_{e,\text{atom}}$ is attractive just as $\mathcal{V}_n$, the sign of $\mathcal{V}^{(2)}_{\text{atom}}$ will be opposite to that of $\mathcal{V}^{(2)}$ in the nuclear case, provided the nuclear potential is predominant. This sign inversion arises from the large Larmor effect in the atomic case: according to (12), (13), one has here $\vec{\omega}_L = -2\vec{\omega}_T$, so that the sign of $\mathcal{V}^{(2)}_{\text{atom}}$ is fixed by $-\vec{\omega}_T$, as against $+\vec{\omega}_T$ in the nuclear case. We conclude that *the order of the doublet terms in an odd-mass nucleus, as a result of the relativistic spin-orbit coupling, will in general be the reverse of that of the one-electron atomic spectra*. In the latter case, the order of the levels would be *regular*: the lower state would have the lower

* See, e.g., DANCOFF and INGLIS [36].

J-value. *In the nuclear spectrum, the doublets should therefore be inverted:* the lower state should have the higher J-value. This rule, enunciated by INGLIS [36], will hold generally if the odd nucleon is a proton; if it is a neutron, it will only hold provided the Thomas effect of nuclear origin is larger than the Larmor effect (of opposite sign) due to the neutron's anomalous magnetic moment.

In atomic theory, it is an easy matter to extend the discussion of multiplet splitting to many-electron spectra (assuming normal coupling conditions): the electrons can be regarded as moving in a central field, and their mutual interaction neglected in the initial approximation. One finds regular or inverted multiplets according as the last orbital shell is less or more than half filled up. On the quasi-atomic nuclear model, we might repeat, *mutatis mutandis*, the same argument. The electrostatic energy $\mathcal{V}_e$ would be much smaller in absolute value than the fictitious central field $\mathcal{V}_n$ representing the bulk of nuclear interaction: Inglis' rule could therefore be generalized for any nuclear multiplets in the simple form: *nuclear multiplets will be inverted or regular according as the last orbital shell is less or more than half filled.* As to the application of this theoretical rule to actual nuclei, however, one should not lose sight of the two very questionable assumptions on which it rests: the idealization of the quasi-atomic model and the assumption that, besides the relativistic spin-orbit coupling, there would be no other non-central coupling of comparable or even larger order of magnitude. In fact, the doubtful validity of either assumption in most cases renders the above rules quite unreliable: they have been mentioned here chiefly to warn the reader against their uncritical use *.

In order to obtain a more general expression for the relativistic interaction between pairs of nucleons, it suffices to discuss the case of two nucleons. The extension of Furry's method to this case leads to simple results when the static nuclear potential does not depend on the spins of the nucleons (Wigner and Heisenberg forces). The corrections $\mathcal{H}^{(2)}$ to be added to the Hamiltonian

$$\mathcal{H}_{\text{free}} = \vec{\alpha}^{(1)}\vec{p}^{(1)} + \vec{\alpha}^{(2)}\vec{p}^{(2)} + (\varrho_3^{(1)} + \varrho_3^{(2)})\, M \tag{15}$$

of a pair of free nucleons (we have put $\vec{\alpha} = \varrho_1\vec{\sigma}$) in order to take account of the interaction in an approximately invariant way (exact to the second order in the velocities) have been derived by BREIT [37a, 38a] for the case that the non-relativistic potential is treated either as the time-component of a 4-vector (we then call it $\mathcal{V}_e$) or as a scalar $\mathcal{V}_n$; he finds

$$\mathcal{H}_e^{(2)} = \qquad \mathcal{V}_e - \tfrac{1}{2}\left[\vec{\alpha}^{(1)}\vec{\alpha}^{(2)} - (\vec{\alpha}^{(1)}\vec{x})(\vec{\alpha}^{(2)}\vec{x})\,\frac{1}{r}\frac{d}{dr}\right]\mathcal{V}_e \tag{16}$$

and

$$\mathcal{H}_n^{(2)} = -\varrho_3^{(1)}\varrho_3^{(2)}\,\mathcal{V}_n - \tfrac{1}{2}\left[\vec{\alpha}^{(1)}\vec{\alpha}^{(2)} + (\vec{\alpha}^{(1)}\vec{x})(\vec{\alpha}^{(2)}\vec{x})\,\frac{1}{r}\frac{d}{dr}\right]\mathcal{V}_n, \tag{17}$$

respectively. The interaction $\mathcal{H}_e^{(2)}$, which is, of course, entirely analogous to

* See, for instance, the comparative estimates of the relativistic spin-orbit coupling of ^{7}Li on the quasi-atomic and α-particle models (*17.53*).

the electromagnetic one, gives rise, when the reduction to large components is carried out, to 3 second order terms of coupling between the nucleons *:

(*a*) a Thomas spin-orbit coupling

$$\frac{\hbar}{(2M)^2}\frac{1}{r}\frac{d\mathcal{V}_e}{dr}\left[(\vec{x}\wedge\vec{p}^{(1)})\vec{\sigma}^{(1)}-(\vec{x}\wedge\vec{p}^{(2)})\vec{\sigma}^{(2)}\right],$$

(*b*) a "Larmor" spin-orbit coupling

$$-2\frac{\hbar}{(2M)^2}\frac{1}{r}\frac{d\mathcal{V}_e}{dr}\left[(\vec{x}\wedge\vec{p}^{(2)})\vec{\sigma}^{(1)}-(\vec{x}\wedge\vec{p}^{(1)})\vec{\sigma}^{(2)}\right],$$

(c) an interaction analogous to that between two magnetic dipoles **

$$\frac{\hbar^2}{(2M)^2}\left\{\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\cdot\tfrac{2}{3}\triangle\mathcal{V}_e+\left[\tfrac{1}{3}\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}-(\vec{x}_0\vec{\sigma}^{(1)})(\vec{x}_0\vec{\sigma}^{(2)})\right]r\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathcal{V}_e}{dr}\right)\right\}.$$

The interaction $\mathcal{H}_n^{(2)}$ gives rise only to a Thomas spin-orbit coupling of opposite sign to that resulting from $\mathcal{H}_e^{(2)}$.

If the nuclear potential involves a spin exchange operator P_σ (Bartlett and Majorana forces), the expression to be added to $\mathcal{H}_{\text{free}}$ becomes more complicated; Breit gives it only for the case that $\mathcal{V}_e P_\sigma$ is treated as the time-component of a 4-vector, in the form

$$\begin{aligned}\mathcal{H}_e^{(2)}=\mathcal{V}_e P_\sigma-\tfrac{1}{2}&\left[\vec{\alpha}^{(1)}\vec{\alpha}^{(2)}-(\vec{x}\vec{\alpha}^{(1)})(\vec{x}\vec{\alpha}^{(2)})\frac{1}{r}\frac{d}{dr}\right]\mathcal{V}_e P_\sigma\\+\frac{i}{2M}&\left[(\vec{\alpha}^{(1)}\vec{x})\frac{1}{r}\frac{d\mathcal{V}_e}{dr}(\vec{x}\wedge\vec{p}^{(12)})\vec{\sigma}^{(2)}-(\vec{x}\wedge\vec{p}^{(12)})\vec{\sigma}^{(1)}(\vec{\alpha}^{(2)}\vec{x})\frac{1}{r}\frac{d\mathcal{V}_e}{dr}\right.\\&\left.+\mathcal{V}_e\vec{\alpha}^{(1)}(\vec{p}^{(12)}\wedge\vec{\sigma}^{(2)})-(\vec{p}^{(12)}\wedge\vec{\sigma}^{(1)})\vec{\alpha}^{(2)}\mathcal{V}_e\right]P_\sigma.\end{aligned}\tag{18}$$

But Breit observes further that Furry's method does not yield the most general expressions compatible with the requirement of approximate invariance. He constructs expressions of the same general form but involving several arbitrary constants; the interactions derived by Furry's method appear as special cases of those general expressions: on account of their origin, they might perhaps be more likely than others. For spin independent nuclear forces, the total spin-orbit coupling can be of the form

$$-\frac{\hbar}{(2M)^2}\frac{1}{r}\frac{d\mathcal{V}}{dr}\left\{[\vec{x}\wedge(a'\vec{p}^{(1)}+(1-a')\vec{p}^{(2)})]\vec{\sigma}^{(1)}-[\vec{x}\wedge(a''\vec{p}^{(2)}+(1-a'')\vec{p}^{(1)})]\vec{\sigma}^{(2)}\right\}$$

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* Besides these terms, there also appear other relativistic corrections involving, for a system of nucleons, the operator $\sum_{i,k}\vec{p}^{(ik)}\operatorname{grad}^{(i)}\mathcal{V}$. Since the latter commutes with the total orbital momentum $\sum_i\vec{x}^{(i)}\wedge\vec{p}^{(i)}$, such terms only give rise to small displacements of the energy eigenvalues, but cause no perturbation of the orbital and spin quantum numbers of the stationary states.

** For $\mathcal{V}_e = 1/r$, it just reduces to the magnetic interaction (*6.13*–20).

with two arbitrary constants a', a''; with $\vec{p}_b = \vec{p}^{(1)} + \vec{p}^{(2)}$, this may be written

$$-\frac{\hbar}{(2M)^2}\frac{1}{r}\frac{d\mathcal{V}}{dr}\left\{(\vec{x}\wedge\vec{p}^{(12)})[(2a'-1)\vec{\sigma}^{(1)}+(2a''-1)\vec{\sigma}^{(2)}]+\tfrac{1}{2}(\vec{x}\wedge\vec{p}_b)(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})\right\}; \quad (19)$$

the above cases of $\mathcal{V}_e$, $\mathcal{V}_n$ correspond to $a' = a'' = -1$ or $+1$, respectively. For a potential of the form $\mathcal{V}P_\sigma$ we can have a spin-orbit coupling consisting of the expression (19) multiplied *on the right* by P_σ and the additional term

$$-\frac{i}{(2M)^2}\mathcal{V}(\vec{p}^{(1)}\wedge\vec{p}^{(2)})(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)})P_\sigma. \quad (20)$$

With reference to table *15.21*, we see that the relativistic couplings may, besides an axial dipole coupling, involve spin-orbit couplings of types III and IV, as well as other couplings of types (20) and

$$\mathcal{M}_4^{(12)} = (\vec{x}\wedge\vec{p}_b)(\vec{\sigma}^{(1)}-\vec{\sigma}^{(2)}); \quad (21)$$

the isotopic factors of these new types of interactions may be any combination of the isotopic factors 1, 2 and 3. The Thomas and "Larmor" terms are respectively proportional to

$$\mathcal{M}^{(12)}+\tfrac{1}{2}\mathcal{M}_4^{(12)} \quad \text{and} \quad \mathcal{M}^{(12)}-\tfrac{1}{2}\mathcal{M}_4^{(12)}. \quad (22)$$

The preceding considerations disclose the general property that *if we neglect second order effects altogether, we have just to use the unrelativistic potential energy without any correction.* As to the order of magnitude of the relativistic corrections, several estimates have been carried out in connexion with the interpretation of the fine structure of certain nuclear levels; we shall come back to this question later (*17.43, 17.53*). At the moment, we shall just mention the results relating to the two-nucleon system. The displacements of the ground state and of the virtual ^1S level of the deuteron due to second order effects have been calculated by SHARE and BREIT (BREIT [37*b*]) on the basis of the expressions (16), (17), (18). For the resulting contribution to the ^1S–^3S separation, they find (assuming a well potential of width d) $0{,}36m$, $0{,}24m$ and $0{,}81m$, respectively. The second order effects to be expected for proton-neutron and proton-proton scattering have been estimated by HOISINGTON [40] and found to be quite small even for energies of 16 MeV.

15.3. Non-central and non-static couplings on meson theory

15.30. Invariance considerations will not give us any information on the distance dependence of the nuclear potential and they leave us the choice between several possible types of dependence on the other coordinates, or combinations of these types. If we want a more precise specification, from a theoretical point of view, it is natural that we should turn our attention to field theories of nuclear interaction. In this section, we shall discuss the forms of nuclear potential which can be derived from meson field theory.

15.31. *Static interactions due to meson fields.* On the hypothesis of weak coupling between nucleons and meson fields, we shall expect the main contribution to the interaction energy of slowly moving nucleons to arise from the static part of the meson field (*8.31*). Each one of the four possible types of meson fields (*1.32*) gives rise to a definite form of static interaction between a pair of nucleons. By way of illustration, we shall outline the derivation in the case of a pseudoscalar field and simply quote the results for the other types *. It will of course suffice to treat the case of a neutral theory.

The equations of the neutral pseudoscalar meson field can be derived from the Hamiltonian

$$\mathcal{H}_\psi = \tfrac{1}{2}\int [\Phi^2 + (\mathrm{grad}\,\Psi - \vec{P})^2 + \varkappa^2\,\Psi^2]\,dv - \int (N\Psi + Q\,\Phi)\,dv, \tag{1}$$

in which Ψ is the field amplitude and Φ its canonical conjugate, obeying the commutation rule

$$[\Phi(\vec{x},t),\ \Psi(\vec{x}',t)] = \frac{\hbar}{i}\,\delta(\vec{x}-\vec{x}'). \tag{2}$$

The quantities N, $\vec{P}$, Q are the source densities, consisting of a pseudoscalar

$$N = \sqrt{4\pi}\,f_1\,n, \qquad n = \sum_i \varrho_2^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)}) \tag{3}$$

and a pseudovector

$$\begin{aligned} \vec{P} &= \sqrt{4\pi}\,\frac{f_2}{\varkappa}\,\vec{p}, \qquad & \vec{p} &= \sum_i \vec{\sigma}^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)}) \\ Q &= \sqrt{4\pi}\,\frac{f_2}{\varkappa}\,q, \qquad & q &= \sum_i \varrho_1^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)}); \end{aligned} \tag{4}$$

the formulae (3), (4) give the representation of the source density operators in the configuration space of the nucleons: the coupling constants f_1, f_2 have the same dimension as an electric charge and play a role entirely analogous to that of the elementary charge e in electromagnetic theory: they are specific parameters attached to a nucleon, which give a measure of the intensity of the interaction between this nucleon and the meson field. An essential difference from the electromagnetic case is that one has here to introduce two independent densities, one of which has the same covariance properties as the meson field itself, while the other transforms like the derivatives of the field. (In electromagnetism, the former is the charge and current density, transforming like the potential; the latter would be an intrinsic density of electric and magnetic polarization.)

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* On this subject, see, e.g., MØLLER and ROSENFELD's paper [40], in which further references are given.

With the help of the commutation rules (2), one gets from the Hamiltonian (1) the field equations

$$\begin{aligned} \dot{\Psi} &= \Phi - Q \\ \dot{\Phi} &= \operatorname{div}(\operatorname{grad} \Psi - \vec{P}) - \varkappa^2 \Psi + N. \end{aligned} \tag{5}$$

Denoting by an index ° the static approximation, and observing that

$$\begin{aligned} \overset{\circ}{Q} &= \overset{\circ}{N} = 0 \\ \overset{\circ}{\vec{P}} &= \vec{P}, \end{aligned} \tag{6}$$

one finds that the static field obeys the equations

$$\begin{aligned} \dot{\overset{\circ}{\Phi}} &= 0 \\ \triangle \overset{\circ}{\Psi} - \varkappa^2 \overset{\circ}{\Psi} &= \operatorname{div} \vec{P}, \end{aligned} \tag{7}$$

with the solution

$$\overset{\circ}{\Psi}(\vec{x}) = -\frac{1}{4\pi} \int \operatorname{div} \vec{P}(\vec{x}') \cdot \varphi(r)\, dv'; \tag{8}$$

in this last formula, $\varphi(r)$ represents the meson potential $e^{-\varkappa r}/r$. The static energy is obtained by inserting (6) and (7) into (1), which gives, after partial integrations,

$$\overset{\circ}{H}_{\mathrm{tr}} = -\tfrac{1}{2} \int \vec{P} \operatorname{grad} \overset{\circ}{\Psi}\, dv; \tag{9}$$

a term $\frac{1}{2}\int \vec{P}^2 dv$ of contact interaction, which would give rise to an unacceptable singularity (*5.132*), has here been left out. This may be justified by the remark that it is always possible, by adding to the Hamiltonian an invariant — in this case $\frac{1}{2}\int (Q^2 - \vec{P}^2)\, dv$ — without influence on the field equations, to reduce any static term of contact interaction to a similar one of the *second* order in the nucleon velocities: now, as we shall see presently in more detail, the meson field theory cannot claim to be reliable up to that order, and all second order non-static effects have to be discarded. In this (admittedly far from elegant) way we can get rid of all unwanted contact interactions.

When the expression (8) for $\overset{\circ}{\Psi}$ is inserted in the static energy (9), the latter takes a form involving only the nucleon variables. This includes in the first place self-energy terms, which are of course infinite when a point model for the nucleons is adopted, as is implied by the δ-functions occurring in the source densities (3), (4). We are only interested in the interaction terms; taking account of these expressions for the source densities, the static interaction energy may be written

$$\overset{\circ}{V}_{\mathrm{tr}} = -\tfrac{1}{2} \sum_{i,k} \left(\frac{f_2}{\varkappa} \right)^2 (\vec{\sigma}^{(i)} \operatorname{grad}^{(i)}) (\vec{\sigma}^{(k)} \operatorname{grad}^{(k)})\, \varphi(r^{(i,k)}). \tag{10}$$

On the symmetrical theory, we would have obtained a similar expression, with additional isotopic factors $\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}$. The form (10) is characteristic of the pseudoscalar theory. Although, e.g., a scalar meson field has a Hamiltonian of exactly the same *form* as (1), the meaning of the source densities is quite different: in contrast to (6), only $\overset{\circ}{\vec{P}}$ vanishes in this case; this explains how one gets an entirely different outcome for the static interaction energy. The accompanying table summarizes the results obtained for

15.31. *Static interactions according to meson field theory*

Types of meson fields		Source intensity constants		Static interaction between a pair of nucleons on neutral theory *
Spin 0	scalar	scalar f_1'	vector f_2'	$-f_1'^2\,\varphi(r)$
	pseudoscalar	ps.-sc. f_1	ps.-vect. f_2	$f_2^2\,\mathcal{S}^{(12)}$
Spin 1	vector	vector g_1	tensor g_2	$[g_1^2 + g_2^2\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}]\,\varphi(r) - g_2^2\,\mathcal{S}^{(12)}$
	pseudovector	ps.-vect. g_1'	ps.-tens. g_2'	$-g_1'^2\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\,\varphi(r) - (g_1'^2 - g_2'^2)\,\mathcal{S}^{(12)}$

* On symmetrical theory, multiply by isotopic factor $\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}$.

the four types of mesons. It will be seen that, besides central forces, only the type of interaction defined by the operator

$$\mathcal{S}^{(12)} = -\frac{1}{\varkappa^2}\,(\vec{\sigma}^{(1)}\,\mathrm{grad}^{(1)})\,(\vec{\sigma}^{(2)}\,\mathrm{grad}^{(2)})\,\varphi(r) \tag{11}$$

occurs.

On account of these results, a *purely* scalar meson theory can readily be dismissed. On a neutral scalar theory, a separation of ^3S and ^1S states of the deuteron could only appear as a non-static perturbation; on a symmetrical theory, the isotopic factor would indeed give rise to different effective potentials for ^3S and ^1S states, but the ^3S potential would be repulsive. In the three other kinds of meson theories, the interaction of type $\mathcal{S}^{(12)}$ occurs effectively; let us therefore look more closely at this operator (ROSENFELD [45*b*]). It can readily be transformed into

$$\begin{aligned}\mathcal{S}^{(12)} &= \tfrac{1}{3}\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\,\varphi(r) + \mathcal{D}^{(12)}\,F(r) - \tfrac{1}{3}\,\Delta^{(12)},\\ F(r) &= \left[1 + \frac{3}{\varkappa r} + \frac{3}{(\varkappa r)^2}\right]\varphi(r)\\ \Delta^{(12)} &= \frac{3}{4\pi\varkappa^2}\lim_{\vec{\xi}\to 0}\int \delta(\vec{x}-\vec{\xi})\,(\vec{\sigma}^{(1)}\vec{\xi})\,(\vec{\sigma}^{(2)}\vec{\xi})\,\xi^{-2}\,d\Omega_{\xi}.\end{aligned} \tag{12}$$

It thus reduces to a combination of central and axial dipole interactions, to which must be added a peculiar contact interaction $\Delta^{(12)}$. By considering the matrix representation of the latter with respect to the stationary states

of the two-nucleon system, one can immediately reduce it to the usual type of contact interaction *

$$\Delta^{(12)} = \frac{1}{\varkappa^2} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \delta(\vec{x}). \tag{12a}$$

The essential point about $\mathcal{S}^{(12)}$ is the distance dependence $F(r)$ of the axial dipole coupling. Here is the critical issue around which the whole discussion about the adequacy of meson field theory for the description of nuclear forces centres. In fact, on account of its pole of the third order at the origin, the potential $F(r)$ cannot be regarded as an exact expression of the law of force at very small distances: for the radial eigenfunction of a $^3P_1$ state of the deuteron, e.g., which is not mixed with any other state (*4.331*), satisfies a wave-equation to which the axial dipole term $\mathfrak{D}^{(12)}F(r)$ contributes an effective potential $\frac{2}{3}F(r)$; such an equation has no regular solution (*5.11*) if the singularity of $F(r)$ at the origin is not reduced to a lower order. Clearly, then, the idealization of point nucleons, which underlies the present formalism of meson theory, fails to provide even an unambiguous derivation of the exact law of force at small distances; and the trouble is that we have no better scheme to guide us in this search. We are therefore driven to the necessity of introducing some arbitrary modification in the theory in order to get rid of the r^{-3} singularity of the non-central coupling: a good deal of the uncertainty still prevailing as regards the true nature of the nuclear forces has its root in this unfortunate circumstance. For there are, of course, several ways of removing the singularity in question and the available evidence does not yet permit us quite definitely to eliminate any of them.

15.32. *Cut-off and mixture.* The procedures followed to free the law of interaction from the r^{-3} singularity belong to two essentially distinct classes, briefly denoted by the epithets "*cut-off*" and "*mixture*". The most radical method consists in *cutting off* the singularity: the validity of the form $F(r)$ for the distance dependence of the non-central interaction is limited to distances larger than some critical radius r_c, while for smaller distances, some other, permissible expression, is substituted for it. Following BETHE [40], who first introduced them, two special forms of cut-off have been discussed most: the *zero cut-off*, according to which the potential is put equal to zero for $r < r_c$, and the *straight cut-off*, in which the constant value $F(r_c)$ is assumed for all $r < r_c$. However, the arbitrary character of the cut-off procedure does not lie so much in the infinite variety of possible assumptions for the substitute potential at small distances **

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* Owing to the radial dependence of the eigenfunctions, the only non-vanishing matrix-elements are those pertaining to transitions between S states, for which the equivalence of the operators (12) and (12*a*) is obvious.

** In all cut-off theories that have been discussed, the cut-off has also been applied to the central part of the interaction, in spite of the fact that there is no cogent reason for doing so.

(this does not affect very much the prediction of the effects of experimental interest), as in the introduction of an additional parameter, the critical radius, the choice of which involves, as we shall see, the gravest difficulties.

The alternative possibility of avoiding the occurrence of the r^{-3} singularity consists in introducing a suitably chosen *mixture* of meson fields of different types, in such a way as to achieve a compensation of the singular contributions due to each of them. More precisely, one assumes that the nucleons are the sources of two or more meson fields, which give independent contributions to the interaction energy. In the most general conception of the "mixture" (or "mixed") theory, suggested by SCHWINGER [42], these independent mesons may even have different masses. The requirement that the total static interaction should not have any third order pole imposes a condition on the intensity constants of the various source densities. For instance, we may consider a mixture of a pseudoscalar and a vector field with respective ranges $\varkappa_0$, $\varkappa_1$, provided the source intensities fulfil the relation

$$\frac{f_2^2}{\varkappa_0^2} = \frac{g_2^2}{\varkappa_1^2}. \tag{13}$$

A glance at table *15.31* will show that many other combinations are possible *a priori*; the determination of the suitable mixture must be completed by different considerations, involving a comparison with the experimental data. The expansion

$$F(r) = \frac{1}{r}\left[\frac{3}{(\varkappa r)^2} - \frac{1}{2} + \frac{3}{2}\varkappa r + \dots\right] \tag{14}$$

further shows that after compensation of the r^{-3} singularity, the remaining one will be a simple pole.

According to Schwinger's general scheme, just outlined, the static interaction of a mixed theory would include both central and axial dipole couplings, just as the cut-off theories. The special form of mixed theory originally proposed by MØLLER and ROSENFELD [40], on the other hand, starts from the simpler assumption of a single value of the meson mass for all kinds of fields entering into the mixture. The elimination of the r^{-3} singularity entails in this case the complete disappearance of non-central interactions. In view of the evidence summed up in *15.1*, this would at first sight appear as an over-simplification. But we must not forget that there are further non-static contributions to the interaction energy, which may be of the required form and magnitude. We shall now turn our attention to these non-static effects, which prove to be of the greatest importance in judging the consistency and suitability of the variants of meson theory just introduced.

15.33. *Non-static interactions due to spin precession and charge exchange.* As already stated (*8.31*), we have to distinguish two kinds of

non-static effects, viz. those depending on the translation velocities of the nucleons, and those arising from the precessions of their spins and isotopic variables. Let us discuss the latter first; we may limit ourselves to the treatment of the spin precession, as that of the charge exchange gives much the same results. Let us consider two nucleons at a fixed distance r apart. In first approximation, the equation of motion of the spin $\vec{\sigma}^{(1)}$ (say) will be

$$\dot{\vec{\sigma}}^{(1)} = \frac{i}{\hbar} [\overset{\circ}{H}_{\psi}, \vec{\sigma}^{(1)}], \tag{15}$$

$\overset{\circ}{H}_{\psi}$ being the static energy of the system. In general, this quantity will involve a spin dependent part of the form

$$G_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi(r) + G_D \mathfrak{D}^{(12)} F(r), \tag{16}$$

the isotopic factor (or rather its average value in the state considered) being included in the coefficients G. On account of (16), equation (15) takes the form

$$\dot{\vec{\sigma}}^{(1)} = \vec{\Omega}^{(1)} \wedge \vec{\sigma}^{(1)}, \tag{17}$$

with

$$\vec{\Omega}^{(1)} = \frac{2}{\hbar} \{ G_\sigma \varphi(r) \vec{\sigma}^{(2)} + G_D F(r) [(\vec{\sigma}^{(2)} \vec{x}_0) \vec{x}_0 - \tfrac{1}{3} \vec{\sigma}^{(2)}] \}. \tag{18}$$

This may be described as a precession of angular velocity (divided by c) $\vec{\Omega}^{(1)}$. Formula (18) shows that for small values of the distance r, the order of magnitude of this angular velocity is $G/\hbar\varkappa^2 r^3$ in a cut-off theory, and $G/\hbar r$ in a mixed theory (G denoting the order of magnitude of the relevant constants G_σ, G_D). This difference has an important bearing on the range of validity of the static approximation in both kinds of theories.

Indeed, it is clear that if the periods of the spin precessions are not large compared with the time of propagation of the main part of the non-static meson fields through a distance of the order of the range of the static nuclear forces, there is no justification in regarding these static forces as a sufficient approximation to the true interaction. This condition defines a critical value R_c of the distance r, such that the validity of the static approximation will be limited to distances larger than R_c. Since the time of propagation of the non-static fields through the distance * $\varkappa^{-1}$ is of the order of magnitude $(\varkappa c)^{-1}$, the critical distance R_c may be defined by

$$\left.\begin{aligned} \varkappa R_c &= (G/\hbar)^{\frac{1}{3}} && \text{on cut-off theory,} \\ \varkappa R_c &= \; G/\hbar && \text{on mixed theory.} \end{aligned}\right\} \tag{19}$$

Since $G/\hbar$, as we shall see, is of the order of 0,1, the limitation imposed

* If several ranges are assumed, $\varkappa^{-1}$ denotes the longest.

by the existence of R_c is much more severe on cut-off than on mixed theory. It even turns out that R_c in the former case is of the same order of magnitude as the cut-off radius itself: this means, indeed, that a cut-off prescription based on the sole consideration of the static potential is essentially inconsistent. *This is in the present state of meson field theory the most serious objection against the cut-off procedure.* It must be conceded that such an objection cannot be regarded as entirely convincing; the advocates of the cut-off idea may in fact point out that the quantum field theory anyway fails to give a reliable account of the forces in the immediate neighbourhood of a particle, and that the cut-off is intended as a rough substitute for the *complete* interaction at small distances which would result from a consistent field theory. Even so, the inconsistency arising from the limitation (19) for cut-off theories enhances their arbitrary and hypothetical character.

On the other hand, while no similar inconsistency occurs in mixed theories, it must be emphasized that such theories are not free from arbitrariness either; only, the element of arbitrariness is introduced, so to speak, in a more subtle way. In fact, the ambiguities of the present formalism of quantum field theory can only be avoided by imposing considerable restrictions on its interpretation. The canonical equations derived from the Hamiltonian are not treated as an exact system of equations, but solved by a process of successive approximations in which, starting from a suitably defined unperturbed system, the calculation of the solution corresponding to a given initial state of this system should not be carried further than the first step leading to a non-vanishing result for the effect under consideration; and such results should of course only be considered as reliable if even this first step does not involve any ambiguity. In quantum electrodynamics, a certain justification for this procedure can be derived from a "correspondence argument", i.e. from the comparison of the quantized problem with the analogous classical calculation; at the same time, the correspondence treatment indicates that the unperturbed Hamiltonian must be taken to include the electrostatic interaction between the charged particles of the system, which can be separated from the non-static part of the electromagnetic field by a canonical transformation. From a purely formal point of view, the general procedure just recalled, including the separation of the static energy by means of a canonical transformation, can be carried over, *mutatis mutandis*, into meson theory (MØLLER and ROSENFELD [40]); but in contrast to the case of electromagnetism it here lacks all physical basis, since there is no empirical evidence on the classical counterpart of the quantized meson fields.

In the last resort, the "correspondence" point of view, thus extended to meson field theory, comes very near to a cut-off prescription, since it also purports to avert the paradoxes of quantum field theory pertaining to the effects at close distance from the sources. In meson theory, such effects arise not only in connexion with the single nucleons (in the form

of self-energy, anomalous magnetic moment and the like), but even in the interaction between them. The mixture method rigorously eliminates the latter difficulty (at any rate from the static interaction) and as regards the former it follows a course entirely analogous to that adopted in quantum electrodynamics. It has, however, to pay for this display of elegance by a greater complexity and lack of cogency, due to the introduction of several kinds of fields with unknown source intensity parameters. The cut-off theory, rough and arbitrary as it is, involves only one type of field, with a smaller number of undetermined parameters.

The limitation of the domain of validity of the law of static interaction symbolized by the critical distance R_c, given by (19), has actually been derived in the spirit of the "correspondence" argument, i.e. by disregarding the quantization of the meson fields. According to the ideas put forward by HEISENBERG [39], other effects, essentially depending on this field quantization, lead to a further, still more restrictive, limitation. This arises from the increase of the probabilities of *explosive processes* in systems of nucleons and mesons, when the energy involved becomes large compared with the rest-mass of the mesons. The corresponding critical length λ_c is the meson wave-length for which the explosive character of the process sets in; one readily finds

$$\varkappa \lambda_c = \sqrt{G/\hbar}. \tag{20}$$

If, as argued by Heisenberg, this critical length has a universal significance, in the sense that the usual concepts of field theory would not be applicable within regions of a linear extension smaller than λ_c, the law of distance dependence of nuclear force derived from any form of meson theory could only claim validity for distances larger than λ_c. Even so, there remains, *on mixed theories*, an interval of distances between λ_c and $\varkappa^{-1}$, in which *the static potential* has a significant value and *can consistently be regarded as representing the main part of the nuclear interaction.*

15.34. *Velocity dependent coupling on mixed meson theory.* It follows from the preceding discussion that on cut-off theory there is no point in taking into account the non-static interaction terms which depend on the translation velocities of the nucleons. On mixed theory, however, such terms may be taken into consideration; they should be treated as perturbations and their contributions retained insofar as they appear as unambiguous results of the calculations. The general expression for the energy operator, exact to the first order in the nucleon velocities, can be derived for each of the four types of meson fields by applying to the original Hamiltonian the canonical transformation which separates the static part of the field; the details of this operation may be found in MØLLER and ROSENFELD's paper [40]. Again, we shall here take as an example the pseudoscalar case and quote the final results obtained in the other cases.

With the notation of *15.31*, the first order interaction energy due to

the pseudoscalar field can simply be written

$$\mathcal{W}_{\psi} = -\int N \mathring{\Psi} dv = \frac{f_1 f_2}{\varkappa} \int \vec{n}(x)\, \vec{p}(x') \text{ grad } \varphi \, dv\, dv', \tag{21}$$

by (3), (4), (8). The velocity dependent matrices ϱ_1, ϱ_2 occurring in the expressions for the source densities can be eliminated by a transformation analogous to the well-known Gordon decomposition of the current density in the Dirac electron theory. For instance, from

$$\frac{d}{c\,dt}\left(\varrho_1^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)})\right) = \frac{i}{\hbar}\,[\varrho_1^{(i)}\vec{\sigma}^{(i)}\vec{p}^{(i)} + \varrho_3^{(i)}M,\ \ \varrho_1^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)})]$$

$$= \text{div}^{(i)}\left(\vec{\sigma}^{(i)}\,\delta(\vec{x}-\vec{x}^{(i)})\right) - \frac{2M}{\hbar}\,\varrho_2^{(i)},$$

we get

$$n(\vec{x}) = -\frac{\hbar}{2M}\,\text{div}\,\vec{p} - \frac{\hbar}{2M}\,\frac{dq}{c\,dt}. \tag{22}$$

In this formula, the last term is of higher order in the velocities and must consistently be neglected; the first term, though formally independent of the nucleon velocities, is nevertheless of the first order, for the divergence operator is equivalent, as to the order of magnitude, to a factor λ^{-1}, λ representing the de Broglie wave-lengths of the nucleons. Other source density operators give rise to a decomposition of the type (22) containing a further term which depends explicitly on the momenta of the nucleons. Inserting (22) in (21) and using the notation (11), we get

$$\mathcal{W}_{\psi} = f_1 f_2\,\frac{M_m}{M}\cdot\tfrac{1}{2}\sum_{i,k}\mathcal{S}^{(ik)}. \tag{23}$$

For the other types of meson fields, the derivation proceeds along the same lines (ROSENFELD [45*b*] *). As in the static case, the final outcomes are very different on account of the different forms of source densities. They are summarized in table *15.34*, in which the same notation has been used as in *15.2*, with in addition

$$\mathcal{M}_5^{(12)} \equiv (\vec{\sigma}^{(1)} \wedge \vec{\sigma}^{(2)})\,(\vec{x} \wedge \vec{p}_b). \tag{24}$$

The table exhibits a striking disparity between the pseudovector case on the one hand and the pseudoscalar and vector cases on the other. The expression pertaining to the former case is not at all of a form which would be expected on general invariance considerations for first-order velocity dependent interactions between a pair of nucleons (*15.21*); in fact, it vanishes for any two-nucleon system whose centre of gravity is at rest.

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* In this paper, however, the treatment of the scalar and pseudovector cases is incomplete; the correct results are given in table *15.34*.

15.34. *Velocity dependent interactions on meson field theory*

Type of meson field		First order interaction between a pair of nucleons on neutral theory*
Spin 0	scalar	$\frac{2 f'_1 f'_2}{\hbar} \frac{M_m}{M} \overline{\vec{p}\,\vec{x}} \frac{\varphi'(r)}{\varkappa^2 r}$
	pseudoscalar	$f_1 f_2 \frac{M_m}{M} \mathcal{S}^{(12)}$
Spin 1	vector	$-g_1 g_2 \frac{M_m}{M} \left\{ (1 + \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}) \left[\varphi(r) - \frac{1}{\varkappa^2} \delta(\vec{x})\right] - \mathcal{S}^{(12)} \right\} - \frac{2 g_1 g_2}{\hbar} \frac{M_m}{M} \frac{\varphi'(r)}{\varkappa^2 r} \mathfrak{M}^{(12)}$
	pseudovector	$\frac{g'_1 g'}{\hbar} \frac{M_m}{M} \frac{\varphi'(r)}{\varkappa^2 r} \mathfrak{M}_5^{(12)}$

* On symmetrical theory, multiply by isotopic factor $\tau^{(1)} \tau^{(2)}$.

The couplings due to pseudo-scalar or vector fields have the expected form and include non-central interactions of both axial dipole and spin-orbit types.

It might at first sight seem disturbing that the distance dependence of the non-static couplings also exhibits a r^{-3} singularity. The purpose of the mixture method, viz. to eliminate such a singularity from the static interaction, would seem to be defeated by its reappearance in the next approximation (FERRETTI [43*b*], HU [45]). At this juncture it is essential to remember that the mixture method is inseparable from the general "correspondence" prescription enunciated above (*15.33*), in virtue of which the non-static interaction cannot be treated as an exact contribution to the Hamiltonian, but only as a perturbation. As such, however, the third order pole does not offer any fundamental difficulty: the situation, in fact, is exactly the same as in the case of a magnetic dipole interaction, touched upon in *6.13*. However, the strong increase of the non-static potential at small r impairs considerably the convergence of perturbation calculations. The situation may be judged by considering the ratio of non-static to static potential; if we assume that all source constants (which enter in different combinations into the two potentials) are of the same order of magnitude, this ratio is of the order $(M_m/M)\ (\varkappa r)^{-2}$. In the significant range of distances $\lambda_c \dots \varkappa^{-1}$, it varies, according to (20), from ≈ 1 to $\approx 0{,}1$: even with the small factor M_m/M, which represents the ratio of mean nucleon velocity to velocity of light for stationary states of nuclei, the non-static contribution to non-central interactions is thus by no means negligible (unless special assumptions are made about the source constants).

15.35. *Spin and life-time of mesons.* The considerations of this section are of a preliminary character. In the next Chapters, we shall have to

confront the predictions of the various forms of meson theories we have been discussing with the properties of nuclear systems, with a view to specifying, if possible, the particular form of theory best adapted to the evidence. But first, we shall enquire in how far cosmic ray data could yield more direct information on the kinds of mesons actually produced in nuclear processes and consequently contributing to the interaction between nucleons. The possibility of detecting mesons of a particular type is, of course, intimately connected with the life-time of these mesons. Theoretically, if we first consider the decay of the meson into a lepton pair, the *relation between its life-time and covariance properties* is determined by the form adopted for the Hamiltonian describing the interaction between the meson and lepton fields. Now, this Hamiltonian, and especially the source densities which govern the production of the meson field by leptons, can be set up in a form identical with that of the meson-nucleon system; the only difference being in the numerical values of the source constants, which will be distinguished from those of table *15.31* by a ˘ mark (e.g. $\breve{g}_2$). A straightforward calculation (SAKATA [41]) leads to the following results for the life-time t_0 of a meson at rest:

$$\begin{aligned}
&\text{scalar:} && \frac{1}{t_0} = c\varkappa\left[\frac{\breve{f}_1'^2}{\hbar} + \frac{\breve{f}_2'^2}{\hbar}\left(\frac{m}{M_m}\right)^2\right] \\
&\text{pseudoscalar:} && \frac{1}{t_0} = c\varkappa\left[\frac{\breve{f}_1}{\sqrt{\hbar}} - \frac{\breve{f}_2}{\sqrt{\hbar}}\frac{m}{M_m}\right]^2 \\
&\text{vector:} && \frac{1}{t_0} = c\varkappa\left[\frac{2}{3}\frac{\breve{g}_1^2}{\hbar} + \frac{1}{3}\frac{\breve{g}_2^2}{\hbar}\right] \\
&\text{pseudovector:} && \frac{1}{t_0} = c\varkappa\left[\frac{2}{3}\frac{\breve{g}_1'^2}{\hbar} + \frac{1}{3}\frac{\breve{g}_2'^2}{\hbar}\right].
\end{aligned} \tag{25}$$

Since any of these formulae contains two undetermined constants, it cannot be directly compared with the experimental results. An additional theoretical aspect has to be adduced; this will be, of course, the conception of the *process of β-decay* proper to meson theory (*1.322*). Let us consider the expression for the life-time of a β-active nucleus to which this theory leads when one assumes that the interaction between nucleons and leptons is brought about by meson fields of a single type; we shall further restrict ourselves to allowed β-transitions of very light nuclei (*A1.11*). The result is of the form corresponding either to Fermi interaction (scalar meson theory), to Gamow-Teller interaction (pseudoscalar and pseudovector meson theories), or to a linear combination of both (vector meson theory). The life-time is thus given by a formula of the type (*A1.11*–1) or the sum of two such expressions; the constants $1/\tau_0$ in the different cases are as follows:

15.35. *Value of β-decay parameter $1/\tau_0$ on meson theory*

Type of field		Fermi interaction term	Gamow-Teller interaction term
Spin 0	scalar	$\frac{8}{\pi}\varkappa c\left(\frac{m}{M_m}\right)^5\left(\frac{f_1'\breve{f}_1'}{\hbar}\right)^2$	—
	ps.-scalar	—	$\frac{8}{3\pi}\varkappa c\left(\frac{m}{M_m}\right)^5\left(\frac{f_2\breve{f}_2}{\hbar}\right)^2$
Spin 1	vector	$\frac{8}{\pi}\varkappa c\left(\frac{m}{M_m}\right)^5\left(\frac{g_1\breve{g}_1}{\hbar}\right)^2$	$\frac{32}{3\pi}\varkappa c\left(\frac{m}{M_m}\right)^5\left(\frac{g_2\breve{g}_2}{\hbar}\right)^2$
	ps.-vect.	—	$\frac{1}{3\pi}\varkappa c\left(\frac{m}{M_m}\right)^5\left[32\left(\frac{g_1'\breve{g}_1'}{\hbar}\right)^2+\left(\frac{g_2'\breve{g}_2'}{\hbar}\right)^2\right]$

The numerical coefficients in this table are those given by SAKATA [41] *; it should be observed, however, that the choice of the Hamiltonian is not unique, owing to the possibility of adding to it terms of direct coupling between nucleons and leptons with arbitrary coefficients: since the available evidence on β-spectra is not sufficiently accurate to allow of a complete determination of these coefficients, we can only say that Sakata's choice is a plausible one, and that the order of magnitude of the $1/\tau_0$-values could only be changed by very improbable assumptions about the direct couplings **. If a mixture of meson fields is adopted, there may appear further interference terms slightly modifying the energy dependence of the life-time (cf. *A1.11*, footnote); this has been thoroughly discussed, on very general assumptions, by ROZENTAL [41a, 45] *** for the case of a pseudoscalar-vector mixture. We shall here disregard this complication, which is not essential for our argument.

The expression for $1/\tau_0$, given by table *15.35*, is more favourable than that for the meson life-time $1/t_0$ to get an insight into the order of magnitude of the lepton source density constants. In the first place, since the empirical evidence is strongly in favour of Gamow-Teller interaction, we infer the unsuitability of a purely scalar theory to account for the phenomenon of β-decay. In any case, the constants $\breve{f}_1'$ and $\breve{g}_1$ must be assumed to vanish, or at any rate to be of a smaller order of magnitude than the others. As regards the latter, if we adopt for $1/\tau_0$ the value (*A1.21*–4), for the meson source density constants (such as f_2, g_2, etc.) the general estimate $g^2/\hbar \approx 0{,}1$ and for the meson mass $M_m \approx 200\,m$, we get for the relevant lepton source constants a rough estimate

$$\frac{\breve{g}^2}{\hbar} \approx 10^{-14} \quad (\breve{g} \equiv \breve{f}_2;\ \breve{g}_2;\ \breve{g}_1',\ \breve{g}_2'). \tag{26}$$

* A misprint in Sakata's paper, affecting the pseudoscalar case, has been corrected.

** On this point, see ROZENTAL [41a].

*** The special choice of parameters advocated in the last paper (ROZENTAL [45]) is questionable.

If we now turn to the formulae (25) for the meson life-time, we see at once that there is an essential difference between mesons of spin 1 and mesons of spin 0. The former would have a life-time of the order of 10^{-9} sec, much shorter than the observed one (*1.332*) of $\approx 2 \cdot 10^{-6}$ sec. *In order to account for β-decay, spin 1 mesons should have a much shorter life-time than those observed at sea-level:* this consequence of meson theory has especially been emphasized by NORDHEIM [39]. On the other hand, the observed life-time of the order of the microsecond can easily be reconciled with the hypothesis of meson fields of spin 0. A scalar field, according to what has just been found, could at most occur in combination with another one and the choice of the constants $\check{f}'_1$, $\check{f}'_2$, is thus indeterminate, the only restriction being the small (or vanishing) value of $\check{f}'_1$. Owing to the peculiar form of the expression of $1/t_0$ in the pseudoscalar case, a value of $\check{f}_2$ of the order of magnitude (26), combined with a suitable choice of $\check{f}_1$ ($\ll \check{f}_2$), will yield a life-time of the required magnitude. Remembering the evidence (*1.333*) pointing to a spin 0 for the sea-level mesons, we come to the conclusion that *the cosmic ray evidence about spin and life-time of mesons as well as the evidence about light β-active nuclei is compatible with a purely pseudoscalar meson theory;* a conclusion first enunciated by ROZENTAL [41*b*].

We may now ask about the bearing of the foregoing discussion on the conception of the mixture theory. Confronting the general results of this discussion with the requirement of obtaining a static potential without third order pole, we immediately see that *any mixture of meson fields,* in order to be compatible with the cosmic ray and β-decay data, *must include mesons of both spins, those of spin 1 having a very short life-time.* If the two kinds of mesons are assumed to have different masses, we have to take account of another type of decay process, by which the heavier meson may go over into the lighter one with emission of a γ-ray. This process involves as an intermediate step the virtual creation of a nucleon pair; it belongs to the same category as that leading to the decay of a neutral meson into photons (*8.311*) and, like this last process, has a very large probability of occurrence. This implies that in a mixed theory involving mesons of different masses, one would have to assume that the heavier ones have spin 1. Taking, e.g., a mixed pseudoscalar and vector meson theory, with a mass ratio 1,6 (*16.41*), the "γ-decay" process just described would lead for the vector mesons to life-times of the order of magnitude of 10^{-18} sec (FINKELSTEIN [47]).

CHAPTER XVI

NON-CENTRAL COUPLINGS AND TWO-NUCLEON SYSTEMS

16.0. The different forms of non-central couplings derived on general grounds in the preceding Chapter must now be tested by comparing their consequences with experiment. In this Chapter, we shall carry out this test insofar as the properties of two-nucleon systems are concerned: we shall review successively the properties of the ground state of the deuteron, the neutron-proton scattering, the proton-proton scattering and the photo-disintegration of the deuteron. First of all, it appears that the only type of non-central coupling having any influence on the behaviour of two-nucleon systems is that we have called axial dipole coupling. We shall first approach the problem from a general point of view, making no assumption on the form of distance dependence of the forces, or schematically representing this dependence by potential wells. Then we shall discuss some variants of meson theories, of both the cut-off and the mixture type.

16.1. General considerations

16.11. *The ${}^3D$-admixture to the ground state of the deuteron.* From the two possible kinds of non-central couplings evolved in the discussion of the foregoing Chapter, viz. the axial dipole coupling $\mathfrak{D}^{(12)}$ and the spin-orbit coupling $\mathfrak{M}^{(12)}$, only the first can contribute to the admixture of ${}^3D$ state to the ${}^3S$ ground state of the deuteron, necessary to account for the electromagnetic properties of this nucleus (*6.12*). Indeed, the other commutes with the orbital momentum and can thus at most give rise to displacements of the levels; it is readily verified that it even vanishes for all singlet states and for ${}^3S$ states *. On the other hand, *the axial dipole operator* $\mathfrak{D}^{(12)}$, which does not commute with the orbital momentum, *is just of the type required to bring about the necessary* ${}^3S + {}^3D$ *combination:* this, of course, follows immediately from the fact that the derivations of $\mathfrak{D}^{(12)}$ as a possible type of interaction and of the ${}^3S + {}^3D$ mixture as the only possible one are based on the same invariance considerations. One can also prove explicitly (*6.121*) that $\mathfrak{D}^{(12)}$ leaves the total angular momentum and the spin multiplicity unaltered, and combines S and D

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* In fact, on account of (*4.32*–16), we may write

$$\mathfrak{M}^{(12)} = -\mathfrak{h}\,[l(l+1) + s(s+1) - j(j+1)]$$

with $s = 0$ for singlet states and $s = 1$ for triplet states.

states. From the easily verified relation

$$\mathfrak{D}^{(12)}\,{}^1(\sigma)_0 = 0, \tag{1}$$

one further concludes that *all singlet states of the two-nucleon system are entirely unaffected by the axial dipole coupling.*

With the help of the formulae (*4.32–17*) for the spin and angle dependence of the eigenfunctions, the matrix-elements of the operator $\mathfrak{D}^{(12)}$ can readily be found (BETHE [40]); the diagonal elements are:

$$\begin{aligned}
(j=l, \quad l\,|\,\mathfrak{D}^{(12)}|\,j=l, \quad l) &= \tfrac{2}{3} \\
(j=l+1,l\,|\,\mathfrak{D}^{(12)}|\,j=l+1,l) &= -\frac{2}{3}\frac{l}{2l+3} = -\frac{2}{3}\frac{j-1}{2j+1} \\
(j=l-1,l\,|\,\mathfrak{D}^{(12)}|\,j=l-1,l) &= -\frac{2}{3}\frac{l+1}{2l-1} = -\frac{2}{3}\frac{j+2}{2j+1};
\end{aligned} \tag{2}$$

the non-diagonal elements, which determine the mixture of states with orbital momenta $j-1$, $j+1$, have the expression

$$(j,l=j\pm 1\,|\,\mathfrak{D}^{(12)}|\,j,l'=j\mp 1) = -2\,\frac{\sqrt{j(j+1)}}{2j+1}\,. \tag{3}$$

Using (2) and (3) we are able to write down explicitly the set of radial equations (*4.331–25*) for all triplet states. The nuclear potential consists of the sum of the central potential (*8.2–1*) $J(r)\,\mathcal{O}$ and an axial dipole potential which we write in the form $F(r)\ \mathsf{I}^{(12)}\,\mathfrak{D}^{(12)}$, including an isotopic factor $\mathsf{I}^{(12)}$. For the latter, we shall chiefly consider either the neutral type $\mathsf{I}^{(12)} = 1$ or the symmetrical type $\mathsf{I}^{(12)} = \boldsymbol{\tau}^{(1)}\,\boldsymbol{\tau}^{(2)}$, and incidentally also the charged type $\mathsf{I}^{(12)} = \mathsf{T}^{(12)}$ (table *15.21*). We thus write

$$\mathcal{V}_{\text{nucl}} = J(r)\,\mathcal{O} + F(r)\ \mathsf{I}^{(12)}\,\mathfrak{D}^{(12)}. \tag{4}$$

In particular, for the ground state of the deuteron, represented by (*6.12–3, 4*), we have to find the two radial eigenfunctions

$$\chi_0(r) = R_0(r)\cos\omega, \qquad \chi_2(r) = R_2(r)\sin\omega \tag{5}$$

as solutions of the system of differential equations

$$\begin{aligned}
\frac{d^2\chi_0}{dr^2} + \frac{M}{\hbar^2}\,[\varepsilon_0 + J(r)]\,\chi_0 &= 2\sqrt{2}\,\frac{M}{\hbar^2}\,F(r)\,\chi_2 \\
\frac{d^2\chi_2}{dr^2} - \frac{6}{r^2}\,\chi_2 + \frac{M}{\hbar^2}\,[\varepsilon_0 + J(r) - 2F(r)]\,\chi_2 &= 2\sqrt{2}\,\frac{M}{\hbar^2}\,F(r)\,\chi_0.
\end{aligned} \tag{6}$$

The *symmetrical* isotopic factor has here been adopted; on a neutral theory, $F(r)$ should replaced by $-\frac{1}{3}F(r)$; if the charged form of isotopic factor is assumed, it should be replaced by $+\frac{1}{3}F(r)$. The sign of $F(r)$ is uniquely related to that of the electric quadrupole moment. The positive sign found

for the latter quantity means, in fact, that the spatial density distribution in the ground state is prolate with respect to the axis of total angular momentum (which is in this case the common direction of the spins). Now, in such a configuration, the mean value of the operator $\mathfrak{D}^{(12)}$ is positive; in order that the potential energy (4) be a minimum, the factor $F(r)\,I^{(12)}$ must be negative. By the same argument, a negative quadrupole moment, or oblate density distribution, would correspond to a positive sign of $F(r)\,I^{(12)}$. We thus get the following correspondence:

		+	−
Sign of quadrupole moment		+	−
Density distribution			
Sign of $F(r)$	Symmetrical or charged type	+	−
	Neutral type	−	+

(7)

16.12. *The axial dipole coupling as a perturbation.* In view of future application to mixed meson theory, we shall first derive some formulae valid on the assumption that the axial dipole coupling can be regarded as a perturbation; we shall only write down the formulae pertaining to a *symmetrical* isotopic factor. If in equations (6) we assume that the terms containing the product $F(r)\,\chi_2$ can be neglected, we may express χ_2 in terms of the eigenfunctions ${}^3R_\varepsilon^{(2)}(r)$ of the continuum states corresponding to the central potential $-J(r)$:

$$\left\{\frac{d^2}{dr^2} - \frac{6}{r^2} + \frac{M}{\hbar^2}\,[\varepsilon + J(r)]\right\} {}^3R_\varepsilon^{(2)}(r) = 0. \tag{8}$$

In fact, if we multiply by ${}^3R_\varepsilon^{(2)*}$ the second equation (6), in which χ_0 on the right may be replaced by R_0, and integrate over r, we get, on account of (8),

$$\int_0^\infty {}^3R_\varepsilon^{(2)*}\,\chi_2\,dr \approx -2\sqrt{2}\,\frac{1}{\varepsilon - \varepsilon_0}\,F_\varepsilon, \tag{9}$$

with

$$F_\varepsilon = \int_0^\infty {}^3R_\varepsilon^{(2)*}(r)\,F(r)\,R_0(r)\,dr.$$

Hence,

$$\chi_2(r) \approx -2\sqrt{2}\int_0^\infty \frac{d\varepsilon}{\varepsilon + |\varepsilon_0|}\,F_\varepsilon\,{}^3R_\varepsilon^{(2)}(r). \tag{10}$$

The normalization condition yields the amount of D-admixture in the form

$$\sin^2 \omega = \int_0^\infty |\chi_2|^2 \, dr = 8 \int_0^\infty \frac{d\varepsilon}{(\varepsilon + |\varepsilon_0|)^2} \, |F_\varepsilon|^2, \tag{11}$$

while, according to (*6.12*–5), the expression for the quadrupole moment reduces in first approximation to

$$\begin{aligned} Q &= -\frac{\sqrt{2}}{10} \, \Re \int_0^\infty R_0^* \chi_2 \, r^2 \, dr \\ &= \frac{2}{5} \int_0^\infty \frac{d\varepsilon}{\varepsilon + |\varepsilon_0|} \, \Re \left[F_\varepsilon \int_0^\infty R_0^* \cdot {}^3R_\varepsilon^{(2)} \, r^2 \, dr \right]. \end{aligned} \tag{12}$$

On the other hand, the correction to the value of the binding energy of the ground state owing to the axial dipole term takes the form, on account of (*6.121*–16),

$$\Delta\varepsilon_0 = \frac{9}{\sqrt{2}} \, \Re \int_0^\infty R_0^* \, F \, \chi_2 \, dr = -18 \int_0^\infty \frac{d\varepsilon}{\varepsilon + |\varepsilon_0|} \, |F_\varepsilon|^2; \tag{13}$$

it is a quantity of the *second* order.

16.13. *Stationary states of not too high energy.* So long as the energy involved in the state or process envisaged is sufficiently small (compared with the unit $\hbar^2\varkappa^2/M$, *5.11*), a general survey of the influence of a non-central coupling can be obtained, as shown by HEPNER and PEIERLS [42], without making any assumption on the form of this coupling. The procedure utilized by these authors consists in neglecting, in all radial integrals, the contributions from the "inside" region, in which the radial wave-functions are distorted by the nuclear potential. In the "outside" region, i.e. for r larger than some distance r', the radial components χ_0, χ_2 can be replaced by the asymptotic expressions

$$\begin{aligned} \chi_0(r) &\simeq e^{-r/a_0} \\ \chi_2(r) &\sim e^{-r/a_0} \left[\left(\frac{a_0}{r} \right)^2 + \frac{a_0}{r} + \frac{1}{3} \right]; \end{aligned} \tag{14}$$

in these formulae, a_0 denotes the characteristic length (*6.21*–14) $a_0 = \hbar/\sqrt{M|\varepsilon_0|}$ directly related to the binding energy. If some plausible value ($\frac{1}{2}$, say) is assumed for the ratio r'/a_0, the amount of D-admixture can be calculated, by the normalization condition, from the observed value of the quadrupole moment, and the eigenfunction of the ground state is then completely fixed in the outside region.

This suffices for a renewed discussion of the *photo-disintegration* of the deuteron (*6.51*) by quanta of not too high energy. The possible admixture of F state in the final P state of the photoelectric process may be disregarded, and the modification of the photoelectric cross-section is found to

be quite negligible. The photomagnetic effect now comprises an additional D–D transition, but its contribution to the cross-section is too small to be observed.

For the treatment of the triplet contribution to the *scattering cross-section of neutrons by protons*, we start from a set of equations analogous to (6), but with a positive energy value ε instead of ε_0. The asymptotic solution is a superposition of S- and D-waves, each consisting of an ingoing plane wave and an outgoing spherical wave. The phases have to be determined from the conditions of continuity to be satisfied at the "boundary" $r = r'$ of the inside and outside regions. The unknown behaviour of the inside wave-functions enters into the result only through three parameters which indirectly represent the influence of the nuclear interactions. One finds that for small values of the energy ε only the S-wave contributes appreciably to the scattering cross-section. The combination of the three parameters entering into the expression for this quantity can finally be reduced, by a procedure generalizing the argument of *5.32*, to the characteristic length a_0 pertaining to the corresponding bound state of relatively low binding energy. The details of the argument will be found in Hepner and Peierls' paper. The outcome of the discussion is again to bring out the insignificant influence of the axial dipole coupling in the energy region considered.

16.2. The Rarita-Schwinger theory

16.20. To proceed further, it will be necessary to make more detailed assumptions on the distance dependence of the nuclear forces. The schematic representation of both the central part $J(r)$ and the non-central part $F(r)$ by wells of suitable widths and depths has the obvious advantage of simplicity and would seem sufficient to give at any rate an idea of the order of magnitude of the various effects investigated. Calculations on this basis have been carried out by RARITA and SCHWINGER [41*a, b, c*]; these authors, however, have introduced an additional simplification, viz. the assumption of wells of equal widths for $J(r)$ and $F(r)$ which detracts to some extent from the generality of the theory. We shall therefore limit ourselves to a brief survey of the main points of interest: for details, the reader may turn to the original papers [4].

[4] Some of the results of Rarita and Schwinger's discussion have recently been extended by MASSEY and HU [47*b*] to other forms of interactions (viz. the exponential, Gauss and meson potential), the essential assumption being to take for both $J(r)$ and $F(r)$ the same form of potential and the same range. In the same category fall some (partly incorrect) calculations by FLÜGGE [39], who took for $J(r)$ and $F(r)$ exponential potentials of equal ranges, and a recent work by MOSZKOWSKI and SACHS [48], using Gauss potentials (unfortunately with an inadequate choice of the range).

It should also be mentioned that results similar to Rarita and Schwinger's had been obtained a little earlier by KLIGMAN [40], but escaped notice owing to their publication in Russian.

16.21. *The ground state of the deuteron.* The eigensolutions of the system (*16.11*–6), when $J(r)$ and $F(r)$ are assumed to be potential wells of the same width D and depths J and $F \equiv \alpha J$, can be calculated explicitly (RARITA and SCHWINGER [41*b*]): outside the well, they have exactly the form (*16.13*–14); the inside part can be adjusted with the help of the continuity conditions, and the remaining amplitudes completely fixed by the normalization condition. The numerical values of the parameters must be chosen so as to account for both the binding energy ε_0 and the electric quadrupole moment Q. For every assumed value of the range D, this fixes the depths J and F.

In particular, for $D = d$, one finds

$$J = 13{,}9 \text{ MeV}, \qquad \alpha = 0{,}775. \tag{1}$$

Comparing this result with that obtained with purely central forces, viz. (table *6.431*–3) $J = 20{,}9$ MeV, we see that — although the amount of D-admixture turns out to be as small as 4 % — the contribution of the non-central force to the binding energy would be very large indeed. In fact, the strength of the central potential for ^3S states is reduced almost to the same value as that for ^1S states (viz. 11,8 MeV), so that practically the whole separation of the ^3S and ^1S levels arises from the non-central coupling. This is the most striking feature of the Rarita-Schwinger theory, — a feature it shares, as we shall see, with most variants of meson theory; for brevity, we shall speak in such cases of a *"large" non-central coupling*.

This situation sharply contrasts with the possibility of a non-central coupling allowing of a perturbation treatment (*16.12*), which will be called the case of *"small" non-central coupling*. It must clearly be expected that the use of the perturbation method would not be allowed for the Rarita-Schwinger potential; indeed, an early calculation by CHRISTY and KUSAKA [39] using this method led, for the same width value $D = d$, to a value of $\alpha \approx 0{,}18$, i.e. to an extreme underestimate of the influence of the axial dipole coupling on the binding energy.

16.22. *Proton-neutron scattering.* For the scattering of neutrons of not too high energy by protons, the accurate computations of RARITA and SCHWINGER [41*b*] of course confirm the general conclusion arrived at by Hepner and Peierls (*16.13*). Compared with the expression derived on the assumption of central forces, the triplet contribution to the scattering cross-section is reduced by a few percent; while this goes towards decreasing the outstanding discrepancy (*6.431*) between the experimental values and those calculated for a well width $D = d$, the correction is too small to be of any avail in judging the reality of this discrepancy.

Interesting new results are obtained in the domain of higher energies of the two-nucleon system, when states of higher orbital momenta, and in the first place the P states, have to be taken into account. A distinction must then be introduced with respect to the dependence on the isotopic

variables, for the properties of the odd states are essentially different for different choices of this dependence (*8.32*). RARITA and SCHWINGER [41c] take a nuclear potential of the general form

$$\mathcal{V}_{\text{nucl}} = 1^{(12)} (a_0 + a_\sigma \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} + a_D \mathcal{D}^{(12)}) \mathcal{J}(r), \tag{2}$$

thus assuming the same isotopic factor for the central and the non-central part of the potential: this assumption appears, of course, quite natural, especially from the point of view of field theory *. They discuss three cases of isotopic dependence: symmetrical, charged and neutral. The relations between the properties of odd-state potentials in these cases are more complicated than those derived in *8.32* for central interactions. Of course, the formulae (*8.32*–18, 19) retain their validity, provided the symbols occurring in them are understood to denote potential *operators*; they may be supplemented by analogous relations pertaining to the charged case:

$${}^\sigma\mathcal{V}^{(\text{odd})}_{\text{charged}} = - {}^\sigma\mathcal{V}^{(\text{even})}_{\text{charged}} \qquad (\sigma = 1, 3; \text{ deuteron}). \tag{3}$$

But the notion of (attractive or repulsive) effective potential cannot be upheld in general.

In the case of P states, however, if one neglects the coupling with F states for $j = 2$, it is still possible to define effective potentials and to study their signs in the different cases of isotopic dependence. These potentials will be different in the three substates ${}^3P_0$, ${}^3P_1$, ${}^3P_2$. The parameters a_0, a_σ, a_D in (2) are fixed by the requirement that the ${}^3S + {}^3D$ and ${}^1S$ potentials have the respective forms

$$\begin{aligned} \mathcal{V}({}^3S + {}^3D) &= -(1 + 3a\,\mathcal{D}^{(12)})\,\mathcal{J}(r) \\ \mathcal{V}({}^1S) &= -q'\,\mathcal{J}(r), \end{aligned} \tag{4}$$

a being the depth ratio introduced above in the discussion of the ground state (*16.21*). This implies, on the one hand,

$$a_0 - 3a_\sigma = -q', \tag{5}$$

on the other,

$$\begin{array}{rccc} & \text{symm.} & \text{charged} & \text{neutral} \\ a_0 + a_\sigma = & \frac{1}{3}, & 1\ , & -1 \\ a_D = & a, & 3a, & -3a. \end{array} \tag{6}$$

If we now write the effective potentials in ${}^3P_j$ states on symmetrical theory in the form

$$\mathcal{V}({}^3P_j)_{\text{symm}} = \tfrac{1}{3}\,{}^3p_j\,\mathcal{J}(r). \tag{7}$$

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* There is a certain similarity between the operator (2) of the symmetrical Rarita-Schwinger theory and that of the corresponding pseudoscalar meson theory (*16.31*–1). In fact, a slight reduction of the value assumed for the range of force to $D = 2.7 \cdot 10^{-13}$ cm causes the constant a_0 to vanish, so that, by formula (6) below, the coefficient a_σ takes the value $\frac{1}{3}$. In this case, according to (5), $q' = 1$, i.e. the binding of the deuteron is entirely due to the axial dipole term.

we have in the first place, on account of (*8.32*–18, 19), (3) and (6),

$$\begin{aligned} \mathcal{V}({}^3P_j)_{\text{charged}} &= {}^3p_j\, J(r) \\ \mathcal{V}({}^3P_j)_{\text{neutral}} &= -{}^3p_j\, J(r). \end{aligned} \tag{7a}$$

Further, with the help of the formulae (*16.11*–2) for the diagonal elements of the operator $\mathfrak{D}^{(12)}$, we find

$${}^3p_0 = -(4a-1), \qquad {}^3p_1 = 1 + 2a, \qquad {}^3p_2 = 1 - \tfrac{2}{5}a. \tag{8}$$

With a value of a of the order of magnitude indicated in (1), we see that ${}^3p_0$ is negative, while ${}^3p_1$ and ${}^3p_2$ are positive. *In the symmetrical or charged Rarita-Schwinger theory,* therefore, *the ${}^3P_0$ potential is attractive, while the ${}^3P_1$ and ${}^3P_2$ potentials are repulsive; in the neutral theory, the situation is reversed.* On account of the smaller statistical weight of the ${}^3P_0$ state, its effect will generally not upset the trend of the two others, so that the situation, so far as no higher than P states are concerned, is qualitatively much the same as that noted in *8.32* for central interactions.

This is indeed apparent in the proton-neutron scattering problem. Numerical computations of the differential cross-section have been carried out for a kinetic energy $\mathsf{E} = 15{,}3$ MeV of the incident neutrons, using the range value $D = d$ and the corresponding parameter values (1), together with the ${}^1S$ strength ${}^1J^{(0)} = 11{,}8$ MeV (i.e. $q' \approx 0{,}85$). A weakness of such calculations, which equally affects those pertaining to meson theories with large axial dipole interaction, presently to be discussed, is that (owing to the prohibitive amount of numerical computation involved) they only include the contributions from P-waves but neglect those from waves of higher orbital momenta. The experience with analogous calculations for central forces (*8.32*) leads us to believe that such a neglect may involve serious errors. Be this as it may, Rarita and Schwinger's results exhibit the same contrast between symmetrical and neutral theory as those of central force theory, even in a still more accentuated manner: the anisotropy ratio $A = \mathcal{S}(\pi)/\mathcal{S}(\frac{1}{2}\pi)$ (*8.32*–23) has the values

$$A_{\text{symm}} = 1{,}16, \qquad A_{\text{neut}} = 0{,}525 \qquad (\mathsf{E} = 15{,}3\ \text{MeV}); \tag{9}$$

the charged theory, with

$$A_{\text{charged}} = 0{,}92, \tag{9a}$$

occupies an intermediate position. The symmetrical theory is quite compatible with the results of the Bristol experiments, which, on the other hand, would lead to the rejection of the neutral theory.

This last conclusion is confirmed in an interesting way by the discussion of the total cross-sections:

$$\begin{aligned} \mathcal{S}_{\text{symm}} &= 0{,}621 \cdot 10^{-24}\ \text{cm}^2 \\ \mathcal{S}_{\text{charged}} &= 0{,}666 \cdot 10^{-24}\ \text{cm}^2 \qquad (\mathsf{E} = 15{,}3\ \text{MeV}) \\ \mathcal{S}_{\text{neut}} &= 0{,}983 \cdot 10^{-24}\ \text{cm}^2. \end{aligned} \tag{10}$$

While the two first figures are of the expected magnitude (*6.413*), the last one shows a considerable deviation from this value, due to the fact that in this case the contribution of the *P*-wave to the *total* cross-section is not small. This contradicts the criterion enunciated in *6.431*. All fast neutron scattering experiments therefore concur to exclude the neutral variant of the Rarita-Schwinger theory; on the other hand, they are compatible with the symmetrical form of the theory.

16.23. *Radiative processes.* For relatively small energies, RARITA and SCHWINGER [41*b*] have verified that no significant evidence regarding the axial dipole coupling can be expected from measurements in this region of either the photo-disintegration of the deuteron or the radiative capture of neutrons by protons. At higher energies, however, the influence of states of higher orbital momenta again makes itself felt in a characteristic way (RARITA, SCHWINGER and NYE [41*a*], RARITA and SCHWINGER [41*c*]). The calculations have been performed for the γ-rays emitted in the reaction $^7Li(p, \gamma)$, of energy $\hbar\nu = 17{,}5$ MeV. For such energies, the photomagnetic contribution, as well as that due to electric quadrupole transitions, is quite negligible on any theory, so that we have essentially to do with the photoelectric effect, due to electric dipole transitions.

The most striking effect of the non-central coupling is to modify the simple $\sin^2\vartheta$-law of angular distribution of the ejected nucleons. This modification arises from the fact (*16.22*) that the radial components of the $^3P_0$, $^3P_1$ and $^3P_2$ final states are affected in different ways by the axial dipole interaction, so that there appears in the differential cross-section, besides the $\sin^2\vartheta$ term, an additional term which is isotropic:

$$d\Phi_{el}(\vartheta) \sim \sin^2\vartheta + \Phi_0. \tag{11}$$

The ratio H (*8.34*) of the intensities of ejected nucleons in directions parallel and perpendicular to that of incidence of the radiation has therefore, in contrast to the case of central forces, a non-vanishing value

$$\mathsf{H} = \frac{\Phi_0}{1 + \Phi_0} \tag{12}$$

even at high energies. For the three cases of isotopic dependence, one finds:

	Φ_0	H
Symmetrical	0,015	0,015
Charged	0,077	0,071
Neutral	0,36	0,265

($\hbar\nu$ 17,5 MeV) (13)

The reason for the large difference between the symmetrical and the neutral theory is to be sought in the predominant influence of the strong $^3P_1$ attraction in the latter theory, which produces a relatively large

perturbation of the corresponding state. Clearly, since the occurrence of the isotropic term Φ_0 is entirely the result of the perturbations of the ^3P states, an appreciable value of this term will then be expected.

On the other hand, the effect of such an attraction in the final P state will be to decrease the matrix-element of the electric dipole if the wave-length of the ejected nucleons is comparable with the radius of the deuteron in its ground state. The *total* photoelectric cross-section will thus be smaller on neutral than on symmetrical theory. Indeed, the computation yields

$$\begin{aligned} \Phi_{\text{symm}} &= 0{,}768 \cdot 10^{-27}\,\text{cm}^2 \\ \Phi_{\text{charged}} &= 0{,}723 \cdot 10^{-27}\,\text{cm}^2 \qquad (\hbar\nu = 17{,}5\ \text{MeV}) \\ \Phi_{\text{neutral}} &= 0{,}376 \cdot 10^{-27}\,\text{cm}^2. \end{aligned} \tag{14}$$

It appears from these results that measurements, with high energy γ-rays, of the total disintegration cross-section and, still better, of the angular distribution of the ejected nucleons could lead to a decision between the neutral and the symmetrical Rarita-Schwinger theory. But if we have to deal with a symmetrical isotopic factor, it would be very difficult indeed to trace the difference between the Rarita-Schwinger interaction, with large non-central coupling, and a purely central potential.

16.3. Meson theories with cut-off

16.30. Just as it is to be regretted that the assumptions of the Rarita-Schwinger theory are too narrow to permit a general survey of the properties of axial dipole couplings, one must deplore the lack of any systematic discussion of meson field theories using some cut-off device to eliminate the singularities of interactions at small distances. The curse of all meson theories, especially those with large non-central coupling, is the amount of laborious numerical computations involved in any problem. As a result of the chaotic situation prevailing among research workers in theoretical physics, the outcome of the active discussions on this topic is a medley of fragmentary information from which it is difficult to gain a general idea, while the gaps, on account of the prohibitive numerical work involved, cannot readily be filled. This section will therefore be nothing more than a very unsatisfactory attempt at piecing together the disconnected results available on the subject. The general conclusion, so far as it goes, is very unfavourable for cut-off theories.

16.31. *Symmetrical pseudoscalar theory.* Among the meson theories using only one definite kind of field, the choice, from general considerations, would seem to be fairly obvious: in order to secure agreement with the observations on the spin of the cosmic ray mesons and to get a non-central term of static interaction, we must adopt the *pseudoscalar* type of meson field. According to the rule (*16.11*–7), the sign of the deuteron quadrupole moment, considering the form of the static interaction given by table *15.31* together with formula (*15.31*–12), further imposes the *symmetrical* com-

bination of charged and neutral fields. The nuclear potential is thus, on this theory,

$$\mathcal{V}_{\text{ps. scal.}} = f_2^2\, \boldsymbol{\tau}^{(1)} \boldsymbol{\tau}^{(2)} \left[\tfrac{1}{3}\, \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}\, \varphi(r) + \mathfrak{D}^{(12)} F(r)\right]. \tag{1}$$

The binding of the ground state of the deuteron is entirely determined by the axial dipole term, and depends therefore sensitively on the cut-off radius. Once a definite value has been assumed for the range of the force, i.e. for the meson mass, the cut-off radius and the source constant are fixed by the binding energy of the deuteron and the slow neutron scattering cross-section. *All other properties of the ground state, in particular the amount of D-admixture and the electric quadrupole moment, then follow unambiguously and allow a test of the theory: this is an attractive feature of any cut-off theory working with only one source constant.*

Unfortunately, when this procedure is carried out (FERRETTI [41*b*]), assuming for the meson mass the (somewhat low) value $M_m = 177\, m$, the cut-off radius turns out to be larger than the range $\varkappa^{-1}$ of the potential. This means, of course, that the specification of the distance dependence by the meson potential becomes illusory. The manner of cutting-off is immaterial; besides the zero and straight cut-off (*15.32*), Ferretti tried the extreme assumption of a very large repulsive potential at small distances, without achieving any substantial improvement.

16.311. *Symmetrical zero spin meson theory.* As a possible way out of this difficulty, FERRETTI [41*b*] suggests an ingenious modification of the theory, which consists in assuming only the neutral mesons to be of the pseudoscalar type, while the charged mesons would be scalar ones; the "symmetrical" character of the theory is upheld, however, by maintaining the variables $\tau_i^{(l)}$ in the expressions for the source densities (*8.31*–5, 10) and assuming the source constants f_2 and f_1 to be equal in absolute value. The resulting nuclear potential is therefore (*8.31*–7)

$$\mathcal{V}_{\text{Ferr}} = f_2^2 \left[\tau_3^{(1)} \tau_3^{(2)} \left(\tfrac{1}{3}\, \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}\, \varphi + \mathfrak{D}^{(12)} F\right) - 2\, \mathrm{T}^{(12)} \varphi\right]; \tag{2}$$

it differs from that of the purely pseudoscalar theory (1) by a term

$$-2\, f_2^2\, \mathrm{T}^{(12)} \left[\left(\tfrac{1}{3}\, \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} + 1\right) \varphi + \mathfrak{D}^{(12)} F\right] \tag{3}$$

which vanishes in ${}^1S$ states; the charge independence property is thus restricted to those states only (*8.1*). In the ground state of the deuteron, the potential due to the scalar field is repulsive; consequently, the axial dipole term must yield a stronger attraction than in the potential (1), and must thus be cut-off at a smaller distance. A reasonable value of the cut-off radius is obtained, as well as the right sign and order of magnitude of the quadrupole moment. Ferretti, however, does not insist further on this theory, which is admittedly rather artificial.

16.32. *Neutral vector theory.* It is somewhat paradoxical that the type of meson field which has most extensively been studied should be just the *vector* field, for no choice could be more unfortunate; not only because it corresponds to a wrong value of the spin, but also because the right sign of the deuteron quadrupole moment, by the same argument as above (*16.31*), can only be given by the *neutral* variant, against which the gravest objections can be raised (*8.31*). Nevertheless, when BETHE [40] published a very thorough investigation of a special form of vector meson theory, this theory was uncritically adopted by various physicists * as the basis of further discussion. We shall here rapidly review the main facts brought to light by this ill-spent labour.

Bethe's potential is that corresponding to a tensor source only ($g_1 = 0$), i.e. [table *15.31* and formula (*15.31*–12)]:

$$\mathcal{V}_{\text{Bethe}} = g_2^2 \, [\tfrac{2}{3} \, \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \varphi - \mathfrak{D}^{(12)} F]. \tag{4}$$

In triplet states, this expression has, qualitatively, the same properties as Ferretti's potential (2) and leads, therefore, to plausible results for the ground state of the deuteron. Bethe adopts the value $M_m = 177\, m$ for the meson mass: since we are dealing with neutral mesons, this choice, though of course inspired by the value then believed to apply to the observed charged mesons, is in effect entirely arbitrary. The cut-off radius depends only slightly on the particular way of cutting-off and turns out to be $\approx 0{,}4$ or $0{,}3$ of the range; further, one finds $g_2^2/\hbar \approx 0{,}08$, corresponding to ${}^1\mathcal{J}^{(0)} \approx 14{,}2$ MeV. The resulting proportion of D state is a little too high (6,6 or 6,8 %), but the quadrupole moment comes out remarkably well.

As regards the effective potentials in ${}^3P$ states, the situation in Bethe's theory is essentially different from the Rarita-Schwinger case (*16.22*): the ${}^3P_1$ potential is attractive, while the other two are repulsive. No general prediction can consequently be made about the anisotropy of the scattering of fast neutrons by protons. This problem has first been treated summarily by BREIT and KITTEL [39*c*] and later very extensively by FERRETTI [41*a*, *b*]. Numerical results are available for an energy $\mathsf{E} = 17{,}4$ MeV of the incident neutrons, account being taken of the P-wave scattering only. In contrast with the Rarita-Schwinger theory, and in agreement with experiment, it is found that the P-waves do not substantially contribute to the *total* scattering cross-section. But the anisotropy

$$A_{\text{Bethe}} = 0{,}545 \qquad (\mathsf{E} = 17{,}4 \text{ MeV}) \tag{5}$$

indicates a strong predominance of forward scattering, quite at variance with the trend suggested by the Bristol experiments.

The *proton-proton scattering* is only slightly affected, at low energies, by the P-wave anomaly (*7.12*). Computations of this anomaly on

* As I mentioned my doubts about Bethe's theory to a distinguished American friend, he dismissed all objections with the remark: "Anyway, it is the latest!".

Bethe's theory have been carried out by BREIT *et al.* [40] and by THAXTON *et al.* [40*b*] for the range 0,67 ... 3 MeV of proton energies. The *P*-wave anomaly is larger than on the assumption of a central potential; in fact, it is too large to agree with observation, unless the meson mass be accorded a much larger value than is assumed in Bethe's theory. The cross-section for the *photo-disintegration of the deuteron*, calculated by RARITA, SCHWINGER and NYE [41*a*] for 17,5 MeV photons, gives the following results, similar to those of Rarita and Schwinger's neutral theory (*16.23*–13, 14):

$$\begin{aligned} \Phi_0 &= 0{,}26, \qquad \mathsf{H} = 0{,}206 \\ \Phi_{\text{Bethe}} &= 0{,}376 \cdot 10^{-27}\,\text{cm}^2. \end{aligned} \qquad (h\nu = 17{,}5\ \text{MeV}) \tag{6}$$

16.33. *Unsymmetrical zero spin meson theory.* At the time that the Italian results about fast neutron scattering (*6.42*) seemed to point to a field theory of the neutral type, an interesting attempt was made by HULTHÉN [44*a*, *b*, 45*b*] to set up a theory of this type, using only scalar and pseudoscalar fields, and taking account in an essential way of the existence of charged mesons. Hulthén assumes that the meson field produced by nucleons is mostly of the *neutral scalar* type, but that a *charged pseudoscalar* field can also be produced with a much smaller intensity. The nuclear potential resulting from these assumptions may therefore be written

$$\mathcal{V}_{\text{Hulth}} = -f_1'^2\, \varphi_0 + 2\mathsf{T}^{(12)} f_2^2 \left[\tfrac{1}{3}\, \vec{\sigma}^{(1)} \vec{\sigma}^{(2)}\, \varphi + \mathfrak{D}^{(12)} F\right], \tag{7}$$

with

$$f_2^2 \ll f_1'^2; \tag{8}$$

the symbol φ_0 indicates that the mass of the scalar meson is not assumed to be identical with that of the pseudoscalar one, so that the range of the potential φ_0 is not necessarily the same as that of φ and F; in fact, Hulthén adopts for the scalar meson the mass value $M_m^0 = 330\,m$ indicated by the proton-proton scattering experiments (*7.13*), while the mass of the pseudoscalar mesons is naturally the observed one. The effective ${}^1S$ potential of the deuteron system differs by $-2f_2^2\varphi$ from that of the like nucleon system $-f_1'^2\varphi_0$: this is qualitatively in accordance with the conclusion drawn from the analysis of the scattering data (*8.1*); the figures of table *8.1* show that the ratio $f_2^2 : f_1'^2$ should not exceed a few percent.

Obviously, in such a theory, the separation between ${}^3S$ and ${}^1S$ states of the deuteron arises from the non-central coupling due to the charged field; owing to the small value of the coupling constant, the cut-off radius must therefore be taken rather small, viz. $\approx |f_2/f_1'|\,\varkappa^{-1}$, which is a satisfactory feature of the theory. On the other hand, a quadrupole moment of the right order of magnitude is obtained, as is verified with the help of the perturbation formula (*16.12*–12). Hulthén's theory predicts a predominantly forward scattering of fast neutrons by protons, in contradiction with the Bristol experiments (HULTHÉN [44*a*]).

16.4. Mixed theories

16.40. Let us finally turn our attention to mixed theories (*15.32*). The alternative here is between theories using mesons of different or equal masses. At first sight, the former possibility seems to afford a simpler picture, since it yields a static axial dipole coupling and one might thus expect to do without the non-static couplings, which, on the contrary, are essential in the case of a mixture of equal mass mesons. From the quantitative point of view, however, the static approximation of Schwinger's mixture proves insufficient. We cannot, therefore, dispense with the introduction of the velocity dependent forces. If, on the other hand, we treat the non-static axial dipole coupling by perturbation theory, we find that, with a reasonable assumption about the order of magnitude of the relevant source constants, it is just sufficient to account for the observed D-admixture of the ground state and its quadrupole moment. It would thus seem that the hypothesis of different masses for spin zero and spin one mesons — as long as no empirical evidence in its favour is forthcoming — is an unnecessary complication: this would bring us back to Møller and Rosenfeld's original proposal of a mixed theory using mesons of uniform mass. However, a closer examination shows that the results of perturbation calculations in this case are very unreliable, so that the assumption of equal masses does not lead to a simpler description of nuclear forces. The present section contains the elaboration of the preceding argument.

16.41. *Schwinger's mixed theory.* The mixed theory proposed by SCHWINGER [42] is of the *symmetrical* type; the nucleons are assumed to give rise to a *pseudoscalar* and a *vector* meson field; the source constants g_2, f_2 are connected by the relation (*15.32*–13) or

$$g_2^2 = f_2^2 \mu^2, \tag{1}$$

where μ is the ratio of the masses of vector and pseudoscalar mesons. Let us further put

$$g_1^2 = f_2^2 \gamma^2 .$$

If $\varkappa$ denotes the inverse range of the pseudoscalar field (and, therefore, $\varkappa\mu$ that of the vector field), the nuclear potential of Schwinger's theory may be written

$$\mathcal{V}_{\mathrm{Schw}} = f_2^2 \boldsymbol{\tau}^{(1)} \boldsymbol{\tau}^{(2)} \left[\tfrac{1}{3} \vec{\sigma}^{(1)} \vec{\sigma}^{(2)} \frac{e^{-\varkappa r} + 2\mu^2 e^{-\varkappa\mu r}}{r} \right. \\ \left. + \gamma^2 \frac{e^{-\varkappa\mu r}}{r} + \mathfrak{D}^{(12)} (F_\varkappa - \mu^2 F_{\varkappa\mu}) \right]. \tag{2}$$

The sign of the deuteron quadrupole moment, considering formula (*15.32*–14), requires

$$\mu > 1. \tag{3}$$

In order to determine the numerical value of μ, JAUCH and HU [44] have calculated, for different choices of this parameter, the value of f_2^2 required to yield, on the one hand, the binding energy of the deuteron, on the other the slow neutron scattering cross-section; they assumed $\gamma = 0$ and took for the mass of the pseudoscalar meson the value $177m$, a little too low according to the most recent estimates (*1.331*). The intersection of the two curves $f_2^2(\mu)$ gives

$$\mu = 1{,}6\,, \qquad f_2^2/\hbar = 0{,}05\,. \tag{4}$$

Unfortunately, for these values of the parameters, the calculated value of the quadrupole moment, using only the static potential (2), comes out about 3 times too small. This, as stated above, spoils the economy of Schwinger's theory. For, instead of putting simply $f_1' = 0$ so as to cancel the velocity dependent coupling altogether, we should choose for this source constant some suitable non-vanishing value, in order to get an additional contribution of the required magnitude to the axial dipole potential. The question then arises whether one could not just take $\mu = 1$ and make the non-static potential responsible for the whole observed deviation from spherical symmetry in the ground state of the deuteron.

The problem of fast neutron scattering has been treated by JAUCH [45] on Schwinger's theory, at any rate so far as P-wave contributions are concerned. He finds an anisotropy ratio

$$A_{\mathrm{Schw}} = 1{,}1 \quad \text{for } E = 14 \text{ MeV}, \tag{5}$$

of much the same order as in the symmetrical theory of Rarita and Schwinger (*16.22-9*). Compared with the results (table *8.33*) due to a purely central meson potential, this figure indicates a tendency of the axial dipole coupling to make the angular distribution of the scattered particles more uniform.

Finally, it might be enquired whether the discrepancy between the value of the mass of the cosmic ray mesons and that deduced from the range of the meson potential required to explain the results of proton-proton scattering experiments (*7.13*), if real, could be brought into harmony with the point of view of Schwinger's theory: it would be taken to indicate the presence of a field corresponding to mesons of larger mass than those observed in cosmic radiation, and which would be identified with the vector mesons postulated by the theory. A closer examination of this point, carried out at my request by Dr. RAMSEY, leads, however, to a negative conclusion. Let us indeed compare the ^1S effective potential derived from the Schwinger interaction operator (2), viz.

$$\mathcal{V}_{\mathrm{Schw}}(^1S) = -f_2^2\,[e^{-r} + (2\mu^2 - \gamma^2)\,e^{-\mu r}]\,\frac{1}{r}\,, \tag{2a}$$

with a simple meson potential of the form

$$\mathcal{V}_{\mathrm{mes}}(^1S) = -G^2\,\frac{e^{-\kappa r}}{r}\,. \tag{2b}$$

If we identify these two expressions and their derivatives for some value r_0 of r, we readily find that the range parameter K is a monotonic function of r_0; the sign of dK/dr_0 being that of $(\gamma^2-2\mu^2)$. Now, since one would expect that the proton-proton scattering is mainly determined by the behaviour of the nuclear interaction at larger distances than the neutron-proton scattering or binding, the trend of empirical evidence, pointing to a larger value of the range parameter for the proton-proton than for the neutron-proton potential, would be interpreted, from the present point of view, to require that K must be an *increasing* function of r_0, i.e., that $\gamma^2 > 2\mu^2$. It is easily seen, however, that this condition, together with $\mu > 1$, would lead to such high values of the coupling constants that the static approximation would break down altogether. Moreover, the meson masses would have to be taken *larger* than the large value suggested by the proton-proton scattering experiments, in contradiction with the cosmic ray data.

16.42. *Møller and Rosenfeld's mixed theory.* The considerable simplification, in mixed theory, implied by the hypothesis of a unique meson mass is that the nuclear interaction is then reduced, in first approximation, to the form of a central potential. The choice of suitable mixtures can be based on the consideration of this static approximation only, as embodied in table *15.31*. An easy discussion (Møller and Rosenfeld [40]), starting from the purely qualitative requirements that (a) the effective potentials in triplet and singlet even states must be attractive, (b) the triplet potential must yield a stronger attraction than the singlet one, leads to only two essentially distinct mixtures compatible with the charge independence property. So far as these requirements go, one may choose either a *symmetrical* type of mixture of a *pseudoscalar* and a *vector* field, or a mixture of *neutral scalar* and *pseudovector* fields. One could add to either mixture further fields of the other types, but most people will feel that a two-field mixture is about as much as they can stand. The evidence about two-nucleon systems has been extensively discussed, from the point of view of central interactions, in Part II of this work. From the values (*6.432–8*) derived from that analysis for the effective potential strengths ${}^3J^{(0)}$ and ${}^1J^{(0)}$ in terms of the meson mass, the expressions for the source constants entering into the static potential are readily obtained. In the case of the symmetrical mixed theory, e.g., for which (table *15.31*)

$$\mathcal{V}_{\mathrm{MR}} = \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)} [g_1^2 + g_2^2 \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}] \varphi(r) \tag{6}$$

with

$$f_2^2 = g_2^2, \tag{7}$$

one has

$$\frac{g_1^2}{\hbar} = \frac{1}{4M_m} ({}^3J^{(0)} - {}^1J^{(0)}), \qquad \frac{g_2^2}{\hbar} = \frac{1}{4M_m} (\tfrac{1}{3}\, {}^3J^{(0)} + {}^1J^{(0)}); \tag{8}$$

the value of

$$\frac{g_1^2}{\hbar} \approx 0{,}03 \tag{9}$$

is approximately independent of the meson mass, whereas that of

$$\frac{g_2^2}{\hbar} \approx 3{,}06 \cdot 10^{-4} \frac{M_m}{m} + 0{,}004 \tag{9a}$$

varies nearly linearly with this mass;

$$\frac{g_2^2}{\hbar} \approx 0{,}065 \qquad \text{for } M_m = 200\, m. \tag{9b}$$

Proceeding to the examination of the velocity dependent potentials (*15.34*), we may at once eliminate the neutral variant (HOLMBERG [44], ROSENFELD [45*b*]), since in that case there would be no combination of S and D states to a first approximation. We are therefore left, as *the only acceptable mixture of two meson fields of equal masses*, with the *symmetrical pseudoscalar-vector theory*. On this theory, the velocity dependent potential, when rid of all contact interactions, consists of additional contributions to the central couplings

$$-\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\frac{M_m}{M}\left[g_1 g_2 + \tfrac{1}{3}(2g_1 g_2 - f_1 f_2)\,\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\right]\varphi(r) \tag{10}$$

and of the non-central interaction

$$\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\frac{M_m}{M}\left[(g_1 g_2 + f_1 f_2)\,\mathfrak{D}^{(12)} F(r) - \frac{2g_1 g_2}{\hbar}\frac{\varphi'(r)}{\varkappa^2 r}\,\mathfrak{M}^{(12)}\right], \tag{11}$$

of which only the first term, representing an axial dipole coupling, needs to be taken into consideration (*16.11*). In view of the factor M_m/M, one is tempted to regard this coupling as "small" (*16.21*), in contrast to the static one occurring in Schwinger's mixed theory. When treated as a perturbation (*16.12*), the interaction (11) only gives rise to a second order displacement of the ground state of the deuteron. At the same time, it yields according to (*16.12*–12) an electric quadrupole moment Q proportional to $g_1 g_2 + f_1 f_2$. Even a rough estimate of Q (MØLLER and ROSENFELD [40], HULTHÉN [43*a*]) shows that the calculated value can be adjusted to fit observation by giving the combination of source constants $g_1 g_2 + f_1 f_2$ some positive value of the same order of magnitude as g_2^2 or g_1^2. This implies that the only effect of the additional central potential (10) will be to modify slightly the determination (7), (8) of the source constants. As a matter of fact, if we also take the experimental value of the quadrupole moment into account, we get instead of these formulae a system of equations of the form

$$\begin{aligned} \frac{g_1^2}{\hbar} - \frac{M_m}{M}\frac{g_1 g_2}{\hbar} &= \alpha \\ \frac{g_2^2}{\hbar} - \frac{1}{3}\frac{M_m}{M}\frac{2g_1 g_2 - f_1 f_2}{\hbar} &= \beta \\ \frac{g_1 g_2 + f_1 f_2}{\hbar} &= \gamma \\ \frac{g_2^2}{\hbar} &= \frac{f_2^2}{\hbar}, \end{aligned} \tag{12}$$

α, β, γ being positive numerical quantities of comparable orders of magnitude. It is easy to see that this system yields two essentially distinct solutions *, according to the sign assumed for the product $g_1 g_2$.

It may be mentioned that the influence of the velocity dependent potential (11) on the scattering of fast neutrons by protons has been summarily discussed by HULTHÉN [43*b*], using Born's approximation. His formulae, however, do not correspond to the velocity dependent potential defined by (10), (11) above, but to a potential including in addition the contact interactions which we have systematically discarded.

16.43. *"Large" or "small" axial dipole coupling.* The conclusion of the preceding subsection would seem to be altogether in favour of the simpler solution afforded by the mixture of meson fields of equal masses. But the argument is essentially based on the application of the perturbation method to the non-static axial dipole coupling; and in view of the rather bad convergence to be expected in this case for perturbation calculations (*15.34*), owing to the peculiar distance dependence of the axial dipole coupling, it may be questioned ** whether the contrast between "large" and "small" axial dipole coupling is really as marked as is suggested by the presentation adopted here for methodical reasons. With a view to settling this question, the numerical integration of the radial wave-equations (*16.11*–6) for the ground state of the deuteron has now been undertaken by Mr. GROSJEAN, assuming in (*16.11*–4) the forms of potentials $J(r)$ and $F(r)$ corresponding to meson theory, viz.

$$J(r) = {}^3J\,\varphi(r), \qquad F(r) = F\,\boldsymbol{F}(r), \tag{13}$$

with arbitrarily fixed ratio of the strengths ${}^3J$ and F, and, of course, some cut-off prescription. Actually, the zero cut-off is adopted, and for each given value of the ratio $F/{}^3J$, the value of ${}^3J$ corresponding to the empirical value of the binding energy is determined as the solution of an

* There subsists, of course, an arbitrariness in the signs corresponding to the invariance of the equations (12) for a simultaneous change of sign of g_1 and g_2, or f_1 and f_2, or both pairs. But this arbitrariness has no physical significance.

** I am indebted to Prof. Peierls for stimulating discussions on this point.

eigenvalue problem; the eigenfunctions χ_0, χ_2 give directly in each case the corresponding amount of D-admixture and electric quadrupole moment.

Although this investigation is not yet completed, the first results already allow some general conclusions to be drawn. In the first place, if we adopt for the ratio

$$\alpha = -\frac{\text{av}\, \mathrm{I}^{(12)} \cdot F}{{}^{3}J} \tag{14}$$

numerical values of the order of 10^{-1}, corresponding to the expression for α,

$$\alpha = \frac{M_m}{M} \cdot \frac{g_1 g_2 + f_1 f_2}{g_1^2 + g_2^2}, \tag{15}$$

given by the Møller-Rosenfeld symmetrical mixed theory, we find much too small values for the admixture of D state and the quadrupole moment: this means that estimates based on perturbation calculations are not to be trusted in this case. In fact, assuming the central part of the potential to give rise to an *attraction* in the ground state, one has to take, in order to arrive at the right orders of magnitude for the two quantities just mentioned, a value of α of the order of $\frac{1}{2}$; in other words one has to consider the axial dipole coupling as "large". This result, combined with the other arguments developed in the present Chapter, would seem to point to the necessity of adopting some mixture of meson fields, since no single type of field will yield the required value of α; moreover, this mixture must be such as to give rise to an axial dipole coupling even in the static approximation. This last requirement rules out the simple case considered by Moller and Rosenfeld; it might be satisfied by a Schwinger mixture, but we have seen that the prospects of such a theory are not very encouraging. It would therefore seem that the original purpose of introducing a mixed theory, viz. to remove the r^{-3} singularity, has to be abandoned, and that we are left in the awkward position of having to adopt a mixed theory (with equal or different meson masses), which also involves a cut-off prescription.

16.5. General conclusion

All theories involving a non-central term of nuclear interaction are adjusted in such a way as to account for the properties of the ground state and the ${}^1S$ virtual level of the deuteron. The more ambitious ones (the cut-off theories) predict such quantities as the quadrupole moment and amount of D-admixture of the ground state. Others (the Rarita-Schwinger and the mixed theories) can only make use of these quantities to fix the values of their adjustable parameters; even so, the possibility of such a determination, leading to reasonable results, affords in some measure a check on the theory.

The various theories that have been put forward differ widely, however, as regards the properties of states of higher orbital momenta, in particular the P states. Their predictions in this respect can be put to test, at least in principle, in the study of the scattering of fast neutrons by protons. Although no obvious general reason can be given for this result, it is found that neutral theories with "large" axial dipole coupling generally predict a predominance of forward scattering ($A < 1$) for neutron energies of the order of 15 MeV, whereas symmetrical theories lead to predominantly backward scattering ($A > 1$) *. Consequently, the Bristol results would condemn neutral theories with "large" axial dipole coupling; on the other hand, they do not allow us to discriminate between several possible forms of symmetrical theories, nor to eliminate the possibility of a neutral theory involving only "small" axial dipole coupling (*8.33*). The removal of the latter uncertainty, however, and with it the full confirmation of the *symmetrical character of the nuclear interaction* requires only an extension of the measurements to slightly higher energies. A further corroboration of this conclusion may be expected from the study of the disintegration of the deuteron by photon or electron impact, either with increased accuracy in the domain of relatively low energies (*8.34*), or for higher energies (*16.23, 16.32*).

Among the variants of meson theory that have been proposed, none is entirely satisfactory. It seems that one should assume some form of *symmetrical mixed theory*, but probably one involving also a cut-off prescription; whether the masses of the different kinds of mesons occurring in the theory should be taken equal or different cannot be decided. In fact, there is hardly any hope of removing these uncertainties of the meson field description of nuclear forces without more information about the properties of mesons, furnished either by a more refined analysis of relevant cosmic ray data, or by the artificial production of mesons. Evidence from ordinary nuclear processes could only be helpful if its accuracy could be substantially increased. It will not be superfluous, in order to complete the picture of the present uncertainties and prospects of nuclear theory, to recall the further alternative which underlies the whole problem of nuclear forces, viz. that between the assumption of strong or weak coupling of the meson fields with their sources (*1.35*). It is true that the strong coupling treatment leads to very grave difficulties; but here also it is the experiment that has to make a final pronouncement, by the detection of isobaric states of nucleons or the disproof of the occurrence of such states.

* HULTHÉN [44*a*] has calculated the differential scattering cross-section on Born's approximation for various theories: symmetrical pseudoscalar theory (*16.31*), symmetrical and neutral vector theories (*16.32*), Schwinger's mixed theory (*16.41*) and his own unsymmetrical theory (*16.33*). For the anisotropy ratio A, this procedure yields an upper limit if $A > 1$, a lower limit if $A < 1$.

CHAPTER XVII

NON-CENTRAL COUPLINGS AND PROPERTIES OF HEAVIER NUCLEI

17.0. The subject of this last Chapter, which closes the circle of our inquiry, is difficult to approach and has received as yet only sporadic treatment. Although potentially capable of yielding valuable and even decisive information about the properties of nuclear forces, the study of heavier nuclei cannot at present give us more than a certain amount of corroborative evidence regarding conclusions drawn from the analysis of the properties of two-nucleon systems. In this respect, the examination of the conditions imposed on the form of the law of nuclear force by the saturation properties of heavy nuclei is especially interesting. Following a course closely parallel to that of Part III, we shall begin with a study of this general problem, and afterwards turn to more specific questions pertaining to light nuclei: we shall successively discuss the influence of non-central couplings on the binding energy and on the fine structure of the ground states of these nuclei.

17.1. Non-central couplings and saturation properties

The occurrence of non-central couplings in the expression of the interaction of a pair of nucleons may be of considerable influence on the saturation properties of nuclear systems bound by such interactions. This important problem has not yet been fully investigated. Two papers by VOLKOFF [42*a*, *b*], following a suggestion of WIGNER and EISENBUD [39*a*] *, discuss the effect of *axial dipole couplings*. The outcome of this discussion is to exclude such theories as involve an extreme predominance of the axial dipole interaction (e.g. Bethe's theory, *16.32*), but no clear-cut conclusion is reached concerning intermediate cases like the Rarita-Schwinger theory (*16.2*).

The gist of the argument consists in showing that a predominant axial dipole coupling will tend to favour collapsed configurations of a heavy nucleus, characterized by large values of the total spin and possibly the neutron excess, and by a shape strongly deviating from spherical symmetry. We prove, to this effect, that in such configurations the contribution of the axial dipole operator (*16.11*)

$$\frac{1}{2}\sum_{i,k}{}' \, \mathsf{I}^{(ik)}\, \mathfrak{D}^{(ik)}\, \mathcal{F}(r^{(ik)}) \tag{1}$$

to the interaction terms of the ordinary type (*11.11*) is attractive: this

* See also BETHE [40], p. 270.

means, indeed, that the resulting binding energy will not exhibit any saturation (*11.13*). In the first place, it is always possible to give the same sign to all contributions from the isotopic factors $\mathsf{I}^{(ik)}$: in a neutral theory, this is automatically the case, irrespective of the neutron or proton character of the constituent nucleons; in a symmetrical theory, we have only to choose an assembly composed exclusively of nucleons of like charge. Consider, further, a configuration in which all the spins are parallel. Since in such a configuration the average value of $\mathfrak{D}^{(12)}$ is $\cos^2\vartheta - \frac{1}{3}$, we can make the total axial dipole interaction of the ordinary type attractive by suitably choosing the spatial distribution of the particle density: according as $F(r^{(ik)})$ av $\mathsf{I}^{(ik)}$ is positive or negative, we have just to take an oblate or a prolate distribution with respect to the total spin axis.

It is true that the kinetic energy will counteract this effect, and one might think that the potential energy would only prevail for systems with larger numbers of nucleons than actual nuclei. The quantitative discussion of this point, carried out by VOLKOFF [42*a*] both in the case of Bethe's neutral vector theory and in the less extreme case of the neutral Rarita-Schwinger theory, clearly shows that the critical mass number, for which the non-saturated attraction begins to dominate, is well within the region of the actual nuclei. Indeed, the collapse of the nuclear system could only be prevented if the character of the interaction were radically to change at very small distances; e.g. HULTHÉN [44*a*] has pointed out that in his unsymmetrical form of theory (*16.33*) the higher approximation terms of interaction between two like nucleons become repulsive at small distances: such a repulsion might lead to a saturation of the nuclear bindings, analogous to that of intermolecular forces in liquids rather than to that of the chemical bonds of the quantal "exchange" type (*2.22*) *.

If, on the other hand, the axial dipole couplings may be regarded as small perturbations, their total effect will not disturb the saturation properties of the binding due to the central interactions. In fact, the latter will be such as to favour configurations with saturated spins (and minimum neutron excess). Now,

$$\sum_{\sigma_z^{(1)'},\,\sigma_z^{(2)'}} v_{\pm}(1)\, v_{\pm}(2)\, \mathfrak{D}^{(12)}\, v_{\pm}(1)\, v_{\pm}(2) = \pm\,(\cos^2\vartheta - \tfrac{1}{3}), \tag{2}$$

the sign on the right-hand side being + or — according as the sign indices of the spin eigenfunctions v are the same or opposite for the two nucleons.

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* It has recently been suggested (FEENBERG and PRIMAKOFF [46]) that the collapsed configurations of heavy nuclei predicted by the theories with "large" axial dipole couplings might actually represent the stablest states of such nuclei, while their more loosely bound saturation configurations would be metastable: such a situation is termed *"conditional" saturation.* Besides axial dipole couplings, non-static forces such as short range, attractive many-body interactions would be liable to bring about conditional saturation. The collapsed forms of nuclei would constitute extremely rare isotopes of known elements, some curious properties of which are described in the note cited.

Consequently, in any spin-saturated configuration the operator (1) of axial dipole coupling will not give rise to any contribution of the ordinary type to the potential energy, since the integrals pertaining to pairs of nucleons with parallel and antiparallel spins will exactly compensate each other. The situation is just the same as with the central interactions involving the spin factor $\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}$, for which such a compensation likewise occurs in spin-saturated configurations (cf. tables *11.11*–1, 2).

17.2. Non-central couplings and binding energies of light nuclei

As soon as we deal with systems of more than two nuclei, the complexity of the nuclear interactions coming into play increases considerably. While in the interaction between two nucleons only the axial dipole coupling shows up, in dealing with more complex nuclear systems we should take into consideration the possible occurrence of spin-orbit couplings and of many-body forces. This aspect of the problem of nuclear forces has hitherto been entirely neglected; its approach is admittedly very uncertain at the present stage. Only one paper bearing on the question of the influence of non-central couplings on the binding energies of light nuclei has been published, by GERJUOY and SCHWINGER [42]. In this paper, the binding energies of ^{3}H and ^{4}He are calculated, by a variational method, on the assumption that the interaction potential has the form (*16.22*–4), given by Rarita and Schwinger's theory for states of even parity (the only ones occurring in this problem): this implies that only the influence of axial dipole forces on the binding energy is discussed (within the scope of the Rarita-Schwinger theory), while the possible occurrence of other interactions is disregarded.

In the case of three- or four-nucleon systems, the occurrence of non-central interactions implies a breakdown of the conservation of not only orbital momentum, as in the deuteron case, but also the spin: we now get, for ^{3}H, a mixture of doublet and quartet states (*14.11*), for ^{4}He a mixture of singlet, triplet and quintet states. The ground state of ^{3}H, e.g., in which $J = \frac{1}{2}$, is of the type $^2S_{\frac{1}{2}} + {}^2P_{\frac{1}{2}} + {}^4P_{\frac{1}{2}} + {}^4D_{\frac{1}{2}}$; the ground state of ^{4}He ($J = 0$) comprises $^1S_0$, $^3P_0$ and $^5D_0$ eigenfunctions. Moreover, the assignment of a definite parity to the eigenfunction does not, in many-nucleon systems, result in any reduction of the admixture of states of different orbital momenta. However, in first approximation, the axial dipole operator will only couple the $^2S_{\frac{1}{2}}$ state of ^{3}H with $^4D_{\frac{1}{2}}$ states and the $^1S_0$ state of ^{4}He with $^5D_0$ states *. Gerjuoy and Schwinger limit themselves to this approximation; adopting the constants (*16.21*–1) derived from the deuteron properties, they find that the D-admixture in the ground

* This is a simple consequence of the fact that the operator $\sum_{i,k} \mathfrak{D}^{(ik)} F(r^{(ik)})$ transforms, when the spin or space coordinates *alone* are rotated, according to a representation of rank 2 of the rotation group.

states of either of the nuclei ^3H and ^4He would be about 4 %, just as in the case of the deuteron. However, while in the latter case the interference of this small amount of *D*-wave with the *S*-wave produces, on the Rarita-Schwinger theory, practically all the binding of the deuteron, the corresponding effect turns out to be much smaller in the heavier nuclei, with the result that the calculated binding energies of these nuclei become much too small, owing to the decrease of the central potential strength as compared with a purely central interaction (*14.2*). The authors' conclusion is accordingly that the assumptions underlying Rarita and Schwinger's theory, and especially that of equal ranges for the central and non-central interactions, are too restrictive to account successfully for the binding energies of the lightest nuclei *.

17.3. Fine structure of nuclear levels

The preceding discussions of this Part IV, while leaving many details unsettled, have at any rate made it clear that in nuclear systems quite appreciable forces are operative to couple the spins and orbital motions of the constituent nucleons. On any nuclear model such couplings will give rise to a *fine structure* of the stationary states of the system, due to the splitting of the multiplet states. Actually, while the order of magnitude of the separations between levels of low excitation in lighter nuclei is generally ** of a few MeV, there are some conspicuous exceptions, in which groups of levels occur, with intervals of the order of several tenths of a MeV only; and one is tempted to recognize in such close groupings the expected multiplet splittings. However, extreme caution is required in attempting identifications of neighbouring levels as members of a multiplet: for, leaving aside the possibility of mere chance coincidence of unrelated states, other types of fine structure than multiplet splitting are suggested by the α-particle model. In view of the present uncertain position regarding both the most adequate model to be applied in each case, and the exact form of the law of nuclear force, we cannot hope, therefore, to reach unambiguous conclusions about fine structures of nuclear levels, but at best to obtain additional arguments concerning the various possibilities.

In the domain of light nuclei, the available empirical material is rather scarce: the only well-studied instances of fine structures of the ground state are those of the related (unstable) nuclei ^5He and ^5Li and of the stable isotope ^7Li. They will be more closely examined in the following sections. Additional indications, in the case of β-active nuclei, might in principle be expected, as explained in *A1.21*, from the observation of composite β-spectra and of concomitant radiative transitions.

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* On the possible bearing on this problem of the recent determination of the magnetic moment of ^3H, see *A 2.251*.

** See the recent survey of the evidence about energy levels of light nuclei by HORNYAK and LAURITSEN [48].

Indirectly connected with the problems of fine structure is the question of the total angular momentum of the ground state of a nucleus (also when no close-lying excited state has been observed). For the value of this quantity is ultimately determined by just the types of coupling responsible for the fine structures. In fact, it can be calculated, for any nucleus, by starting from a definite model and a definite type of non-central coupling: if the angular momentum has been observed, this provides a check on the assumptions made; if not, it gives us an indication, unfortunately very unreliable, of the value that might be expected. Examples of this kind of argument will be found in *A2.252, A2.253, A2.26*, in connexion with the calculation of magnetic moments; we shall just mention the interesting case of ^{8}Li, whose configuration, on the quasi-atomic model, is characterized by the presence of a half-filled neutron p shell besides a p-proton (ROSE and BETHE [37]).

We shall now proceed to give an account of the two best known cases mentioned above: the five-nucleon systems ^{5}He and ^{5}Li, and the nucleus ^{7}Li.

17.4. Fine structure of ^{5}He and ^{5}Li

17.40. Since no stable state exists for the nuclei ^{5}He and ^{5}Li, the main source of information about their "virtual" levels is the study of nuclear reactions involving one of these nuclei either as a partner in the final stage of the reaction or as the compound nucleus. In the reaction ^{7}Li(d, α), ^{5}He is obtained in a state unstable with respect to neutron emission, which is very probably the lowest one, with an energy of about 0,8 MeV above that of α-particle plus neutron. A sensitive method of investigating the properties of this ground state of ^{5}He is the study of the scattering of neutrons by ^{4}He: the existence of the virtual level of the compound nucleus ^{5}He will be marked by a resonance effect in the dependence of the scattering cross-section on the energy of the impinging neutrons. As we shall see, it is this effect that has actually disclosed the doublet structure of the ground state. The properties of ^{5}Li may serve as a check on the conclusions arrived at with regard to ^{5}He, since the ground states of the two nuclei, according to the symmetry of nuclear forces with respect to charge (*3.3*), should only differ by the Coulomb energy [a], which would raise the ground state of ^{5}Li about 1 MeV above that of ^{5}He: hence the interest of experiments on the scattering of protons by ^{4}He. In this section, we shall therefore begin by establishing the necessary "dispersion formulae" which give the scattering cross-section of Helium for nucleons in terms of the energy of impact. We shall then survey the results of the

[a] An estimate of the Coulomb energy difference between ^{5}Li and ^{5}He, based on Bethe's theory of the "last" nucleon binding (*3.3*), has been carried out by STEPHENS [40] and yields 0.88 MeV. By equating this quantity to the expression $\frac{3}{5}\frac{e^2}{R}(A-1)$, valid for a homogeneous charge distribution of radius R, one gets an "equivalent" nuclear radius $R \approx 4{\cdot}10^{-13}$ cm.

scattering experiments, and finally proceed to a theoretical discussion of the fine structure they reveal.

17.41. *Theory of the scattering of nucleons by Helium.* In investigating the elastic scattering of a nucleon by any nucleus of mass A and charge Z, the latter may to a first approximation be treated just as a point, its structure entering only indirectly through the phase-shifts of the asymptotic wave-functions of the scattered nucleon. Such a treatment neglects the polarization of the nucleus by the impinging nucleon, as well as the effect of the exchange of this nucleon with one of the constituent particles of the nucleus *. The case of proton scattering is the more general one; as for proton-proton scattering (*7.11*), except that here no complication arises from the exclusion principle, and that if we assume the scattering nucleus (A, Z) to have no angular momentum, we are dealing with doublet states of the total system, rather than with the singlet and triplet configurations of the two-proton system.

If we consider the scattering process in the barycentric system of reference, we have to use the reduced mass $M_{\text{red}} = M \frac{A}{A+1}$, and the connexion between the relative kinetic energy ε of the colliding particles, their relative velocity v, relative momentum p and wave-number k, is as usual

$$p = \hbar k, \qquad \frac{v}{c} = \frac{p}{M_{\text{red}}}, \qquad \varepsilon = \frac{p^2}{2\, M_{\text{red}}}. \tag{1}$$

The scattering angle ϑ in the barycentric system is simply related to the angle of recoil Θ' of the nucleus which is at rest in the laboratory system by the relation (*6.212*–20) $\Theta' = \frac{1}{2}(\pi - \vartheta)$. The transformation of scattering angle ϑ and energy ε to the corresponding quantities Θ, E in the laboratory system is different according as the nucleus (A, Z) or the proton is supposed to be at rest in the latter system (scattering of protons or of α-particles). In the former case, we have

$$\varepsilon = \mathsf{E} \cdot \frac{A}{A+1} \tag{2}$$

and the formulae quoted in the first footnote of *14.12*; in the latter case, one has just to replace A by $1/A$ in these formulae. Finally, the characteristic parameter α (*7.11*–2) becomes

$$\alpha = \frac{Z e^2 c}{\hbar v}; \tag{3}$$

with this new meaning of α, the formulae (*7.11*–3, 4, 5, 6) for pure Coulomb scattering are applicable to our case.

Denoting, as usual, by v_+, v_- the spin eigenfunctions of the scattered nucleon, the spin and angular dependence of the doublet states of our

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* Dr. Fröhlich and Dr. Humblet have independently pointed out to me that both effects may be important, especially in the case of scattering by Helium.

system may be written in the form

$$
\begin{aligned}
{}^2Z_{l+\frac12}^{(l)m} &= \frac{1}{\sqrt{2l+1}}\,[\sqrt{l+m+\tfrac12}\,v_+\,Y_l^{m-\frac12} - \sqrt{l-m+\tfrac12}\,v_-\,Y_l^{m+\frac12}] \\
{}^2Z_{l-\frac12}^{(l)m} &= \frac{1}{\sqrt{2l+1}}\,[\sqrt{l-m+\tfrac12}\,v_+\,Y_l^{m-\frac12} + \sqrt{l+m+\tfrac12}\,v_-\,Y_l^{m+\frac12}].
\end{aligned} \tag{4}
$$

The general solution is therefore

$$\Psi = \sum_{l,j} \mathfrak{A}_j^{(l)m} \cdot {}^2Z_j^{(l)m} \cdot \frac{1}{r}\, L_l(r), \tag{5}$$

the asymptotic form (*17.11*–11) of the radial functions L_l depending on the phase-shifts $\delta_j^{(l)}$ due to the proper nuclear interactions; in general, these phases will be different for the two substates $j = l \pm \frac12$ corresponding to a given l. The $\mathfrak{A}$'s are determined by comparing the expression (5) with the solution [cf. (*17.11*–5)]

$$\Psi_c = v_+\,\Psi_c^{(+)} \tag{6}$$

corresponding to pure Coulomb scattering, and by expressing that $\Psi - \Psi_c$ asymptotically represents an outgoing spherical wave; this gives for the scattered wave

$$
\begin{aligned}
\Psi_{\text{scatt}} \sim{}& v_+\,(\Psi_c^{(+)})_{\text{scatt}} \\
&+ \frac{e^{ikr - i\alpha\log 2kr}}{kr} \sum_l e^{2i\eta^{(l)}} \Bigg[\left(\frac{l+1}{\sqrt{2l+1}}\,A_{l+\frac12}^{(l)} + \frac{l}{\sqrt{2l+1}}\,A_{l-\frac12}^{(l)}\right) v_+ \sqrt{4\pi}\, Y_l^0 \\
&\qquad - \sqrt{\frac{l(l+1)}{2l+1}}\,(A_{l+\frac12}^{(l)} - A_{l-\frac12}^{(l)})\, v_- \sqrt{4\pi}\, Y_l^1\Bigg],
\end{aligned} \tag{7}
$$

where

$$A_j^{(l)} = \frac{1}{2i}\,(e^{2i\delta_j^{(l)}} - 1) = e^{i\delta_j^{(l)}} \sin \delta_j^{(l)} \tag{8}$$

and

$$(\Psi_c^{+})_{\text{scatt}} = -\frac{\alpha}{2kr\sin^2\vartheta/2}\, e^{ikr - i\alpha\log 2kr\sin^2\vartheta/2 + 2i\eta^{(0)}}. \tag{9}$$

The existence of virtual levels of the compound system is expressed by the energy dependence of the phase-shifts $\delta_j^{(l)}$ in the following way. Suppose that for some value L of l, there is a virtual level with the doublet structure corresponding to $J = L \pm \frac12$; let $\varepsilon_J^{(L)}$ be the energy values, $\Gamma_J^{(L)}$ the widths of the two virtual substates. Upon the assumption that these widths are sufficiently small (compared with a suitably defined critical energy), Bloch [40] has derived a general dispersion formula, from which

it follows that the phases $\delta_J^{(L)}$ have the form of a smoothly varying function of the energy, due to the so-called "potential scattering" (this we shall continue to denote by $\delta_J^{(L)}$), to which is added a resonance term *

$$^*\delta_J^{(L)} = \operatorname{arctg} \frac{\frac{1}{2}\Gamma_J^{(L)}}{\varepsilon_J^{(L)} - \varepsilon}; \tag{10}$$

this means that the characteristic phase function $\Delta_J^{(L)}$, defined by (8), must be replaced by

$$\begin{aligned} &\Delta_J^{(L)} + e^{2i\delta_J^{(L)}}\, {}^*\Delta_J^{(L)}, \\ &{}^*\Delta_J^{(L)} = \frac{\frac{1}{2}\Gamma_J^{(L)}}{\varepsilon_J^{(L)} - \varepsilon - \frac{1}{2} i \Gamma_J^{(L)}}. \end{aligned} \tag{11}$$

In view of the possible influence of non-central couplings, we must, however, as emphasized by Beck and Tsien [42], envisage a somewhat more general description of the virtual states, involving a mixture of the two states of different orbital momenta $L = J \pm \frac{1}{2}$ which correspond to a definite value of the total angular momentum J. The two substates Ψ_ϱ ($\varrho = \mathrm{I}, \mathrm{II}$) pertaining to a given J will conveniently be represented by the orthogonal eigenfunctions

$$\Psi_\varrho = \sum_{L=J\pm\frac{1}{2}} c_{\varrho L}\, i^{L-(J-\frac{1}{2})} \cdot {}^2Z_J^{(L)m} \cdot \frac{1}{r} L_L(r), \tag{12}$$

with a unitary matrix $c_{\varrho L}$ which may be written

$$(c_{\varrho L}) = \begin{pmatrix} \cos\omega & \sin\omega\, e^{i\chi} \\ \sin\omega\, e^{-i\chi} & -\cos\omega \end{pmatrix}. \tag{13}$$

A straightforward extension of the preceding argument, again starting from Bloch's general dispersion formula, shows that in the expression (7) for the scattered wave, we have to replace the $\Delta_J^{(L)}$ by

$$\begin{aligned} &\Delta_J^{(L)} + e^{2i\delta_J^{(L)}} \sum_\varrho A_\varrho^{(L)} c_{\varrho L}\, {}^*\Delta_J^{(\varrho)}, \\ &{}^*\Delta_J^{(\varrho)} = \frac{\frac{1}{2}\Gamma_J^{(\varrho)}}{\varepsilon_J^{(\varrho)} - \varepsilon - \frac{1}{2} i \Gamma_J^{(\varrho)}}, \\ &A_\varrho^{(L)} = \sum_{L'=J\mp\frac{1}{2}} c^*_{L'\varrho}\, e^{i[\delta_J^{(L')} - \delta_J^{(L)}] + i[\gamma_J^{(L')} - \gamma_J^{(L)}]}. \end{aligned} \tag{14}$$

Let us now write down the differential scattering cross-section in different cases of varying complexity. For an energy value ε of the relative kinetic energy not in the vicinity of any resonance energy, we may use directly the expression (7), in which the phases then practically reduce

* Formula (30) of Bloch's paper, giving this resonance term, is erroneous: but the correct expression is used throughout the paper.

to the potential scattering contributions; we get accordingly, with some obvious transformations *,

$$dS = \frac{d\Omega}{k^2}\,[\,|F_+(\vartheta)|^2 + \sin^2\vartheta\,|F_-(\vartheta)|^2],$$

$$F_+(\vartheta) = \frac{\alpha}{2\sin^2\vartheta/2}\,e^{-i\alpha\log\sin^2\vartheta/2} - \sum_l e^{2i\sum_{\nu=1}^{l}\operatorname{arctg}\alpha/\nu}\,[(l+1)\,\Delta^{(l)}_{l+\frac12} + l\,\Delta^{(l)}_{l-\frac12}]\,P_l(\cos\vartheta) \tag{15}$$

$$F_-(\vartheta) = \sum_l e^{2i\sum_{\nu=1}^{l}\operatorname{arctg}\alpha/\nu}\,[\Delta^{(l)}_{l+\frac12} - \Delta^{(l)}_{l-\frac12}]\,\frac{d}{d(\cos\vartheta)}\,P_l(\cos\vartheta).$$

For pure Coulomb scattering, we have, using (1) and (3), the Rutherford formula

$$dS_c = \left(\frac{Ze^2}{4\varepsilon}\right)^2 \frac{d\Omega}{\sin^4\vartheta/2}. \tag{16}$$

The ratio of the scattering cross-section (15) to pure Coulomb scattering may be written

$$\frac{dS}{dS_c} = |f_+(\vartheta)|^2 + \frac{4\sin^4\vartheta/2}{\alpha^2}\,\sin^2\vartheta\,|F_-(\vartheta)|^2$$

$$f_+(\vartheta) = 1 - \frac{2\sin^2\vartheta/2}{\alpha}\,e^{i\alpha\log\sin^2\vartheta/2}\sum_l e^{2i\sum_{\nu=1}^{l}\operatorname{arctg}\alpha/\nu}\,[(l+1)\,\Delta^{(l)}_{l+\frac12} + l\,\Delta^{(l)}_{l-\frac12}]\,P_l(\cos\vartheta). \tag{17}$$

In the case of neutron scattering, we have

$$dS = \frac{d\Omega}{k^2}\,[\,|F^0_+(\vartheta)|^2 + \sin^2\vartheta\,|F^0_-(\vartheta)|^2]$$

$$F^0_+(\vartheta) = \sum_l [(l+1)\,\Delta^{(l)}_{l+\frac12} + l\,\Delta^{(l)}_{l-\frac12}]\,P_l(\cos\vartheta) \tag{18}$$

$$F^0_-(\vartheta) = \sum_l [\Delta^{(l)}_{l+\frac12} - \Delta^{(l)}_{l-\frac12}]\,\frac{d}{d(\cos\vartheta)}\,P_l(\cos\vartheta).$$

In order to take account of a virtual state consisting of two neighbouring levels $^2P_{\frac12}$, $^2P_{\frac32}$, we should add to the F's in (15) the resonance amplitudes

$$\begin{aligned} R_+(\vartheta) &= e^{2i\operatorname{arctg}\alpha}\,[2\,e^{2i\delta^{(1)}_{\frac32}}\,{}^*\Delta^{(1)}_{\frac32} + e^{2i\delta^{(1)}_{\frac12}}\,{}^*\Delta^{(0)}_{\frac12}]\cos\vartheta \\ R_-(\vartheta) &= e^{2i\operatorname{arctg}\alpha}\,[\,e^{2i\delta^{(1)}_{\frac32}}\,{}^*\Delta^{(1)}_{\frac32} - e^{2i\delta^{(1)}_{\frac12}}\,{}^*\Delta^{(1)}_{\frac12}]\,; \end{aligned} \tag{19}$$

e.g., neglecting all potential scattering except the S-contribution, we get for the ^2P resonance scattering of neutrons

$$dS = \frac{d\Omega}{k^2}\,[\,|e^{i\delta^{(0)}}\sin\delta^{(0)} + (2\,{}^*\Delta^{(1)}_{\frac32} + {}^*\Delta^{(1)}_{\frac12})\cos\vartheta\,|^2 + |\,{}^*\Delta^{(1)}_{\frac32} - {}^*\Delta^{(1)}_{\frac12}|^2\sin^2\vartheta]. \tag{20}$$

If, however, we have a "mixed" resonance of the type (12), (13), (14),

---

* See, in particular, the first footnote of *13.11*.

we must take

$$
\begin{aligned}
R_+(\vartheta) &= (J+\tfrac{1}{2}) \sum_{\varrho} {}^*\!\Delta_J^{(\varrho)} \sum_{L=J\pm\frac{1}{2}} e^{2i\delta_J^{(L)}} A_\varrho^{(L)} c_{\varrho L} P_L(\cos\vartheta) \\
R_-(\vartheta) &= \sum_{\varrho} {}^*\!\Delta_L^{(\varrho)} \sum_{L=J\pm\frac{1}{2}} e^{2i\delta_J^{(L)}} A_\varrho^{(L)} c_{\varrho L} (-1)^{L-(J-\frac{1}{2})} \frac{d}{d(\cos\vartheta)} P_L(\cos\vartheta).
\end{aligned} \tag{21}
$$

In particular, if we may, in first approximation, disregard all potential and Coulomb scattering (the latter becomes small at large scattering angles ϑ) and if the separation of the two p substates is sufficiently large, we get a simple estimate of the resonance cross-section:

$$
\begin{aligned}
d\mathcal{S}_\varrho &= \frac{d\Omega}{k^2} \,|{}^*\!\Delta_J^{(\varrho)}|^2 \,|A_\varrho|^2 \,[\,|R_+(\vartheta)|^2 + \sin^2\vartheta \,|R_-(\vartheta)|^2] \\
A_\varrho &\approx \cos\omega \pm \sin\omega\, e^{i\chi} \\
R_+(\vartheta) &\approx [(l+1)(P_l \cos\omega \pm P_{l+1} \sin\omega\, e^{i\chi})]_{l=J-\frac{1}{2}} \\
R_-(\vartheta) &\approx [P_l' \cos\omega \mp P_{l+1}' \sin\omega\, e^{i\chi}]_{l=J-\frac{1}{2}},
\end{aligned} \tag{22}
$$

i.e., making use of a well-known relation between Legendre functions,

$$
\begin{aligned}
d\mathcal{S}_\varrho &\approx \frac{d\Omega}{k^2} \,|{}^*\!\Delta_J^{(\varrho)}|^2 \,[1 \pm \sin 2\omega \cos\chi] \cdot [S(\vartheta) \pm \sin 2\omega \cos\chi \, S'(\vartheta)] \\
S(\vartheta) &= [(l+1)^2 P_l^2 + \sin^2\vartheta\, (P_l')^2]_{l=J-\frac{1}{2}} \\
S'(\vartheta) &= [(l+1)^2 P_l P_{l+1} - \sin^2\vartheta\, P_l' P_{l+1}']_{l=J-\frac{1}{2}}.
\end{aligned} \tag{23}
$$

The last relation takes an especially simple form in the case of backward scattering, i.e. for $\vartheta = \pi$. From the well-known behaviour of the Legendre polynomials, one derives immediately

$$
\left(\frac{d\mathcal{S}_\varrho}{d\Omega}\right)_{\vartheta=\pi} \approx \frac{1}{k^2} \,|{}^*\!\Delta_J^{(\varrho)}|^2 (J+\tfrac{1}{2})^2 \,[1 - \sin^2 2\omega \cos^2\chi]. \tag{24}
$$

On this estimate of the backward scattering cross-section is based a procedure, the principle of which goes back to P. WENZEL [34], to ascertain the angular momentum and even the degree of mixture, of a resonance level (BECK and TSIEN [42]). In fact, putting $|{}^*\!\Delta_J^{(\varrho)}|^2 = 1$ for the resonance maximum, we obtain from (24), by varying both J and the mixture parameter $a^2 \equiv \sin^2 2\omega \cos^2\chi$, a family of curves representing the maximum backward cross-section in terms of the energy of the impinging particles for different J's and a^2's: the curve on which the observed maximum falls gives us a reliable determination of the J-value and an approximate indication of the value of a^2; from the latter, we can only deduce an upper and a lower limit for the proportion of admixture $\sin^2\omega$ of the substate $L = J + \frac{1}{2}$:

$$
\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-a^2} < \sin^2\omega < \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-a^2}. \tag{25}
$$

17.42. *Analysis of scattering experiments.* The scattering of neutrons by Helium has been investigated in the domain of energies of the impinging neutrons extending from the thermal region up to about 3 MeV; the most interesting feature is the occurrence of a resonance at about 1 MeV: this corresponds to an energy of about $\frac{4}{5} \cdot 1$ MeV or 0,8 MeV for the resonance level, in nice agreement with the energy of the virtual ground state as derived from the ^7Li (d, α) reaction (*17.40*). The cross-section for thermal neutrons, determined by CARROLL and DUNNING [38], is

$$\mathfrak{S}_{\text{therm}} = 1{,}5 \cdot 10^{-24}\,\text{cm}^2. \tag{26}$$

By the method previously described (*6.42*), BARSCHALL and KANNER [40] studied the angular distribution of the recoil α-particles on impact of neutrons of 2,5 MeV and 3,1 MeV energy. A careful study of the backward scattering ($\vartheta = \pi \pm 24^\circ$) in the interesting energy region $\approx$ 0,5 ... 2,5 MeV has been made by STAUB and TATEL [40], using a cloud chamber method; by suitably filling the chamber, they obtained the He-scattering cross-sections relative to those of hydrogen, and took for the latter the values given by the theoretical formula (*6.431–2*) (with a well width d). If Wenzel's criterion, in the form (24) as just explained, is applied to their results, the maximum of the backward scattering cross-section at $\approx$ 1 MeV is found to indicate a nearly pure ${}^2P_{\frac{3}{2}}$ resonance level.

This maximum cross-section, however, exhibits a fine structure, the main peak at 1 MeV neutron energy being accompanied by a somewhat lower and apparently broader one at $\approx$ 1,35 MeV neutron energy. In this case, Wenzel's criterion in the simple form (24) would not be applicable. Staub and Tatel try to analyze their experimental curve for the backward scattering cross-section in terms of neutron energy on the assumption of a doublet resonance level of the type ${}^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$. In using formula (20), which covers this case, one is chiefly hampered by the uncertainty prevailing with respect to the sign and magnitude of the potential scattering phase $\delta^{(0)}$. A clue to the order of magnitude of $\sin^2 \delta^{(0)}$ is provided by the value (26) of $\mathfrak{S}_{\text{therm}}$, but there would seem to be no basis for any conjecture about the sign* of $\delta^{(0)}$. According to the choice of this sign, there are two equally acceptable solutions, corresponding to different orders of the doublet levels:

(27)

Sign of $\delta^{(0)}$ (assumed)	$\delta^{(0)}$ (approxim.)	${}^2\varepsilon^{(1)}_{\frac{3}{2}}$ MeV	${}^2\varepsilon^{(1)}_{\frac{1}{2}}$ MeV	${}^2\varepsilon^{(1)}_{\frac{3}{2}} - {}^2\varepsilon^{(1)}_{\frac{1}{2}}$ MeV	${}^2\Gamma^{(1)}_{\frac{1}{2},\frac{3}{2}}$ MeV	Order of levels
+	+24,9°	0,76	1,08	0,32 ± 0,10	0,32	inverted
—	—37,8°	1,08	0,84	0,24 ± 0,08	0,32	regular

* Staub and Tatel's remarks on this point at the end of their paper are not substantiated by the work of Wheeler and Barschall, mentioned below.

More recent experiments by HALL and KOONTZ [47] fail to bring any further elucidation. These authors use as scattering chamber a He-filled proportional counter which allows them to determine the distribution of the recoil He nuclei in terms of the recoil energy E'. Since the latter quantity is proportional to $\cos^2\Theta'$, i.e. to $1 - \cos\vartheta$, this distribution curve has the same shape as that giving the differential cross-section in terms of $\cos\vartheta$. The distribution curves obtained for neutron energies in the range 0,8 ... 1,6 MeV have the parabolic shape expected, according to formula (20), for P scattering. Unfortunately, the parabolic fits are not sufficiently precise to lead to a unique determination of the parameters characterizing the resonance levels. One can only say that the new data are not incompatible with the above values indicated by Staub and Tatel.

It must be noted that the other available evidence casts some doubt on Staub and Tatel's interpretation of the resonance doublet at 1 MeV neutron energy. In the first place, this interpretation can be tested with the help of Barschall and Kanner's above-mentioned measurement of the differential scattering cross-section for neutrons of 2,5 MeV energy. WHEELER and BARSCHALL [40] attempted to analyze these data by using graphical methods of phase-shift determination similar to that applied by Wheeler to He–He scattering (*13.11*). For such an energy value, far from any resonance region, we may start from the simple expression (18) for the differential cross-section, in which only the terms corresponding to the first few values of l are retained. If we try to fit the empirical curve by a superposition of the ${}^2S_{\frac{1}{2}}$, ${}^2P_{\frac{1}{2}}$ and ${}^2P_{\frac{3}{2}}$ waves only, we find values of the P-phases differing by as much as 60°. This is a much larger difference than can be accounted for by the narrow resonance doublet at 1 MeV: for the eigenfunctions of doublet components of such small separation would be very similar and their phases much nearer equality. This discrepancy might be taken to indicate a considerable contribution from D-waves; it is indeed found that ${}^2D_{\frac{3}{2}}$ and ${}^2D_{\frac{5}{2}}$ waves might be present with a phase difference of about 40°, but it is not possible to estimate with any accuracy the resulting modification of the P-phases.

Another check on Staub and Tatel's results is provided by the experiments on scattering of protons by Helium, carried out especially for this purpose by HEYDENBURG and RAMSEY [41], with the apparatus represented by fig. *7.12*–1. The resonance of the ^{5}Li compound nucleus expected (*17.40*) in the neighbourhood of 2 MeV kinetic energy of the incident protons * was actually found, but the maximum of the scattering cross-section at $\Theta = 140°$ (nearest approach to backward scattering) was flatter and broader than in the case of the ^{5}He system and did not exhibit any fine structure. The application of Wenzel's criterion (TSIEN [44]) yields $J = \frac{3}{2}$ and $0{,}07 < \sin^2\omega < 0{,}93$, which is compatible with the description

* Heydenburg and Ramsey's experiments cover the range 1,2 ... 3 MeV of proton energies. Data for 4,2 MeV protons are supplied by POWELL *et al.* [47*b*].

of the resonance level as a nearly pure $^2P_{\frac{3}{2}}$ state. A closer analysis in terms of $^2S_{\frac{1}{2}}$, $^2P_{\frac{1}{2}}$ and $^2P_{\frac{3}{2}}$ waves has been attempted by LANDAU and SMORODINSKY [44]: they state that the phases of the P-components are found to differ strongly from each other *.

On the whole, it would seem that Staub and Tatel's interpretation of the ^{5}He resonance level, while in serious need of further check, may provisionally be retained, at any rate as a first approximation. It will possibly turn out, on closer study **, that we shall have to deal with mixtures of the two P levels with $^2S_{\frac{1}{2}}$ and $^2D_{\frac{3}{2}}$ states, respectively; and since such admixtures give rise to *two* states of each of the types $^2S_{\frac{1}{2}} + {}^2P_{\frac{1}{2}}$, $^2P_{\frac{3}{2}} + {}^2D_{\frac{3}{2}}$, it may be asked whether the observed fine structure, instead of being primarily the P-doublet splitting, would not rather correspond to either one of these pairs of mixed states. Wenzel's criterion is useful in eliminating such possibilities: for it shows, on the one hand, that the resonance levels cannot both have the angular momentum $J = \frac{1}{2}$, and on the other hand, that the degree of admixture associated with the value $J = \frac{3}{2}$ is too small to be compatible with the occurrence of a close pair of states with this quantum number. In conclusion, we shall confine ourselves, in the theoretical discussion which follows, to the consideration of a simple P-doublet splitting.

17.43. *Fine structure of ^{5}He and spin-orbit coupling.* The properties of the deuteron give us information on the axial dipole interaction between nucleons only, but cannot tell us anything about the possible occurrence of a spin-orbit coupling (*16.11*). The ^{5}He fine structure, on the other hand, if interpreted as a P-doublet splitting, should primarily depend just on the latter coupling; herein lies the special interest of this case. In fact, it is easily seen *** that with a non-central interaction of the axial dipole type a separation of the P-doublet components will be obtained only as a second order perturbation effect. This effect has been discussed by DANCOFF [40]. Let ε_0 be the energy of the unsplit ground state; the displacements $\Delta\varepsilon_{0,J}$ of the substates with $J = \frac{1}{2}, \frac{3}{2}$ due to the axial dipole operator $\mathcal{W} = \sum_{i,k} \mathfrak{D}^{(ik)} F(r^{(ik)}) \mathrm{I}^{(ik)}$ are of the form

$$\Delta\varepsilon_{0,J} = \sum_n \frac{(0,J\,|\,\mathcal{W}\,|\,n)\,(n\,|\,\mathcal{W}\,|\,0,J)}{\varepsilon_0 - \varepsilon_n}, \tag{28}$$

* As regards the ^{5}Li nucleus, the issue is further obscured by the conflicting results of the experiments on scattering of α-particles in hydrogen. Some authors, most recently TSIEN [40, 44] (see also BECK and TSIEN [42]), claim to have disclosed a complex resonance at *lower* energies than Heydenburg and Ramsey, in fact Tsien describes as many as six equidistant resonance levels of ^{5}Li between 0,56 and 1 MeV. Such a structure would not only be very hard to understand by itself, but it could not be reconciled with the empirical and theoretical (*14.21*) evidence tending to show that the virtual state of 0,8 MeV energy of ^{5}He is the lowest one.

** Especially in view of the existence of a ^2S virtual level close to the ^2P state, as revealed by Tyrrell's work (*14.21*).

*** Cf. the footnote of *17.2*.

the summation extending to all intermediate states, of energy ε_n, which give rise to non-vanishing matrix-elements $(0,J|\mathcal{W}|n)$ coupling them to the substates $0, J$ of the ground level. Since the intermediate states involve configurations in which the "α-cluster" of the ground state is broken up, the excitation energies ε_n are at least of the order of 15 MeV. In order to estimate the difference $\Delta\varepsilon_{0,\frac{1}{2}}-\Delta\varepsilon_{0,\frac{3}{2}}$ Dancoff starts from the identity

$$\frac{1}{\varepsilon_0-\varepsilon_n}=\frac{1}{\varepsilon_0-\bar{\varepsilon}}+\frac{\varepsilon_n-\bar{\varepsilon}}{(\varepsilon_0-\bar{\varepsilon})^2}+\frac{(\varepsilon_n-\bar{\varepsilon})^2}{(\varepsilon_0-\bar{\varepsilon})^2\,(\varepsilon_0-\varepsilon_n)}, \tag{29}$$

which, inserted in (28), yields

$$\Delta\varepsilon_{0,J}=\frac{(0,J|\mathcal{W}^2|0,J)}{\varepsilon_0-\bar{\varepsilon}}\left(1-\frac{\bar{\varepsilon}}{\varepsilon_0-\bar{\varepsilon}}\right)+\frac{(0,J|\mathcal{W}\mathcal{H}\mathcal{W}|0,J)}{(\varepsilon_0-\bar{\varepsilon})^2}, \tag{30}$$

provided the mean energy $\bar{\varepsilon}$ is determined by the condition that the contribution to $\Delta\varepsilon_{0,J}$ arising from the third term on the right-hand side of (29) vanishes. In (30), $\mathcal{H}$ represents the unperturbed Hamiltonian. A closer consideration shows that the expectation values $(0,J|\mathcal{W}^2|0,J)$ of $\mathcal{W}^2$ in the two substates will differ very little from each other; in fact, taking for $\mathcal{W}$ the Rarita-Schwinger form * with the constants determined by the deuteron properties (*16.21*), Dancoff estimates that the contribution of the first term of (30) to the doublet splitting will hardly exceed 0,01 MeV. Likewise, the contribution to the splitting from the part of the operator $\mathcal{W}\mathcal{H}\mathcal{W}$ involving the unperturbed *potential* energy will be very small. The *kinetic* part of $\mathcal{H}$, on the other hand, leads to a significant term in $\mathcal{W}\mathcal{H}\mathcal{W}$ (i.e. one having different expectation values in the two substates), which is of the form

$$\frac{1}{2}\sum_{i,k}\frac{\hbar^2}{M}\left[\frac{1}{r^{(ik)}}\,F(r^{(ik)})\,\mathbf{l}^{(ik)}\right]^2\mathcal{M}^{(ik)}, \tag{31}$$

with the spin-orbit operator $\mathcal{M}^{(ik)}$ (type III of table *15.21*). The effect of the axial dipole coupling is thus reduced to that of an "equivalent spin-orbit coupling". The resulting order of the doublet components is seen to be *regular*. As to the order of magnitude of the separation, it is very sensitive to the form of distance dependence $F(r)$ adopted; Dancoff's estimates vary from 0,014 MeV to 0,13 MeV, the latter estimate corresponding to the potential of Bethe's neutral meson theory (*16.32*). Even in this rather extreme case, the axial dipole coupling would thus prove to be insufficient.

There is, of course, an interaction of the spin-orbit coupling type which is always present in any nuclear system: it is the relativistic coupling (of the second order in the nucleon velocities) discussed in *15.22*. Let us estimate the importance of this effect for the ^{5}He doublet. For this purpose,

* Dancoff adopts the neutral theory; but another choice would not essentially modify his estimates.

it will suffice — again following DANCOFF [40] — to consider the odd neutron as moving in the nuclear field $\mathcal{V}_n$ of the α-particle; neglecting the Larmor effect due to the neutron's magnetic moment, we are left with a Thomas coupling of the form (*15.22–11*)

$$-\left(\frac{\hbar}{2M}\right)^2 \frac{1}{r}\frac{d\mathcal{V}_n}{dr} \cdot 2\vec{S}\vec{L}. \tag{32}$$

The expectation values of $2\vec{S}\vec{L}$ for the substates $J = L \pm \frac{1}{2}$ are $+L$ and $-(L+1)$, respectively. Since av $\left(\frac{1}{r}\frac{d\mathcal{V}_n}{dr}\right)$ is obviously positive, the operator (32) gives rise to an *inverted* doublet, with a splitting

$$|{}^2\varepsilon^{(1)}_{\frac{3}{2}} - {}^2\varepsilon^{(1)}_{\frac{1}{2}}| = \frac{3}{4}\left(\frac{\hbar}{M}\right)^2 \text{av}\left(\frac{1}{r}\frac{d\mathcal{V}_n}{dr}\right). \tag{33}$$

If we assume for $\mathcal{V}_n$ the simple form of a well of depth J_n and width D, we get

$$\text{av}\left[\frac{1}{r}\frac{d\mathcal{V}_n}{dr}\right] = \frac{J_n}{D}[R^{(1)}(D)]^2, \tag{34}$$

where $R^{(1)}(r)$ denotes the radial wave-function of the ground state. For the purpose of our estimate, we may take the critical case of no binding energy; with the notations

$$z = \frac{1}{\hbar}\sqrt{2MJ}\cdot r \quad , \quad z_D = \frac{1}{\hbar}\sqrt{2MJ_n}\cdot D, \tag{35}$$

this corresponds to the radial equation

$$\begin{aligned} R'' - \frac{2R}{z^2} + w(z)\,R &= 0 \\ w(z) &= 1 \text{ for } z \leqslant z_D \\ &= 0 \text{ for } z > z_D, \end{aligned} \tag{36}$$

with the solution

$$\begin{aligned} R^{(1)} &\sim \sqrt{z}\, J_{\frac{3}{2}}(z) && (z \leqslant z_D) \\ &\sim z^{-1} && (z > z_D). \end{aligned} \tag{37}$$

The conditions of continuity for $z = z_D$ lead in the first place to

$$\frac{d}{dz}\log[z^{\frac{3}{2}} J_{\frac{3}{2}}(z)] = 0 \quad \text{for } z = z_D,$$

i.e. to $J_{\frac{1}{2}}(z_D) = 0$, or

$$z_D = \pi, \tag{38}$$

which, by (35), fixes the critical well depth in terms of its width. In the

second place, combining the continuity conditions with the condition of normalization, we readily obtain

$$[R^{(1)}(D)]^2 = \frac{2}{3D}. \tag{39}$$

Inserting (34), (39) into (33) and replacing J_n by its value deduced from (35), (38), we finally get

$$|{}^2\varepsilon^{(1)}_{\frac{3}{2}} - {}^2\varepsilon^{(1)}_{\frac{1}{2}}| = \frac{\pi^2}{4}\left(\frac{\hbar}{DM}\right)^4 M. \tag{40}$$

Taking, e.g. $D \approx 4 \cdot 10^{-13}$ cm (as suggested by the equivalent nuclear radius, cf. footnote of *17.40*), equation (40) gives a doublet splitting of about 0,02 MeV, one order of magnitude too small *.

Clearly, therefore, we need some larger spin-orbit coupling of other origin. Now, it will be recalled that a *mixed meson theory* of nuclear interactions may provide such a coupling. In fact (*16.42*–11), it is found that the non-static interaction of the *first* order in the nucleon velocities contains a spin-orbit term, which may be written

$$-\frac{1}{2}\sum_{i,k}\frac{\hbar}{MM_m}\cdot\frac{1}{r^{(ik)}}\cdot 2g_1g_2\,\varphi'(r^{(ik)})\,\mathfrak{l}^{(ik)}\,\mathfrak{M}^{(ik)}. \tag{41}$$

Irrespective of the question as to the values of the masses of the mesons of different spins, this term is seen to arise from the *vector meson field* alone. It is only present when both the vector and the tensor source of this field contribute to its production; and if the source constants g_1, g_2 are of the same order of magnitude, a comparison of the expression (41) with that derived from (*15.22*–19, 20) for the Thomas coupling corresponding to the static meson potential (which involves g_1^2 and g_2^2 instead of g_1g_2) shows that the coupling (41) should produce effects roughly M/M_m larger than the latter, which is just the required order of magnitude. The order of the levels would thus be conditioned by the sign of g_1g_2. From this point of view, the occurrence of a rather large spin-orbit coupling in ^{5}He may be regarded as an indication of the *existence of mesons of spin one*, not yet detected in cosmic radiation (*15.35*).

* A more exact treatment of the relativistic effect, taking the interactions between all pairs of nucleons into account, would have to start from the general expressions (*15.22*–19, 20) for the second order spin-orbit coupling between a pair of nucleons. We shall discuss this treatment in the case of ^{7}Li (*17.5*). For ^{5}He, the calculation has been carried out under the same conditions by KITTEL [42], but in this case, the inclusion in the odd neutron eigenfunction of a radial factor involving an exponential exp $(-r^2/R_\nu^2)$ represents a very bad approximation in view of the virtual character of the ground state. Since the resulting expression for the doublet splitting is very sensitive to the value of the "mean distance" R_ν of the neutron from the centre of gravity of the α-cluster, it is unfortunately of little significance.

17.5. Fine structure of ^{7}Li

17.51. *Empirical data.* An excited state of about 0,45 MeV energy of the ^{7}Li nucleus has been produced in a variety of ways. The results of the most important experiments are collected in table *17.51* below, in which the diverse reactions employed have been classified according to the compound nucleus (*9.11*) whose disintegration leads, in one or two steps, either to the excited or to the normal state of ^{7}Li. We shall first give some brief comment on these nuclear reactions:

(1) The reactions ^{6}Li (d, n) and ^{10}B (p, α) both lead to an unstable ^{7}Be nucleus, which transforms by K-capture either into the normal state of ^{7}Li, or into an excited state of this nucleus, the energy of which is directly given by that of the γ-ray which accompanies the decay process.

(2) Likewise, the production of the nucleus ^{7}Li in two different states by the reactions ^{6}Li (d, p) and ^{10}B (n, α), ^{9}Be (d, α) is revealed by the occurrence of two groups of protons or α-particles of different energies: the difference of these energies yields the energy of the excited state of ^{7}Li. In the case of the ^{6}Li (d, n) reaction, a check on this interpretation has been obtained by the study of the γ-radiation emitted in the course of the reaction: it was found to consist of quanta corresponding, within the limits of the available accuracy, to the transition of the excited ^{7}Li nuclei to their normal state.

(3) Finally, the inelastic scattering of protons or α-particles by ^{7}Li leaves the nucleus in an excited state, the energy of which is again determined by the measurement of the accompanying γ-radiation.

17.52. *Discussion of data.* The first question to be examined is whether the various experimental approaches just summarized all lead to the same excited level, or whether we are in reality dealing with different levels of neighbouring energies:

(1) It is fairly certain that both the neutron decay of the compound nucleus ^{8}Be and the α-decay of ^{11}C give rise to the same state — in fact, the ground state — of the ^{7}Be nucleus, with a definite life-time with respect to transmutation into ^{7}Li. The two modes of K-capture effecting this transmutation correspond to a definite fine structure of the ^{7}Li ground state, with very nearly 0,45 MeV separation. The various measurements on ^{7}Be prepared by deuteron bombardment of ^{6}Li show that within a wide range of excitation energies of the ^{8}Be compound nucleus *, the same state of the ^{7}Be nucleus is reached by the α-decay of this compound nucleus. One might still desire a more accurate measurement of the γ-ray from ^{7}Be prepared by the ^{10}B (p, α) reaction.

(2) The measurements pertaining to the proton decay of the ^{8}Be compound nucleus may most readily be interpreted as involving a definite

* An idea of the excitation of the compound nuclei may be gained from the kinetic energies of the impinging particles, indicated in the second column of table *17.51*.

fine structure of the ^{7}Li residual nucleus, identical with that disclosed by the ^{7}Be decay. The close coincidence of the accurately determined excitation energies derived from the proton groups of the ^{6}Li (d, n) reaction and the γ-ray due to inelastic proton scattering on the one hand, and from the ^{7}Be decay on the other, speak strongly for this identification.

(3) As regards the ^{7}Li fine structure connected with the α-decay of the ^{11}B compound nucleus, the situation is less clear, especially in view of the discrepancy between Miss Graves' α-particle group measurements and the precise γ-ray determination by Siegbahn and Slätis. However, the concordance of the latter result with those of the two first groups of experiments again points to the identification of the ^{7}Li fine structures met with in the various cases.

Taking it for granted, therefore, that all the available data pertain to the same fine structure of the ground state of ^{7}Li, we may proceed to inquire about the possible character of the two-component stationary states. The first interpretation that suggests itself is, of course, that we are dealing with a P doublet (which would be inverted, since the angular momentum of the ground state of ^{7}Li is known to be $\frac{3}{2}$); this would indeed be in keeping with the quasi-atomic model (*10.21*). The properties of the K-capture decay of ^{7}Be would not be incompatible with this description (*A1.21*); but the variation of the relative yield of excited ^{7}Li nuclei with the excitation energy of the compound nucleus ^{8}Be or ^{11}B definitely forces us to abandon it: for, if the two states under discussion — apart from a very small energy difference — are only distinguished from each other by different values of the quantum number J, one would expect the numbers of decays of the compound nuclei leading to these states to be in the constant ratio of the statistical weights $2J + 1$, i.e. $^7\text{Li}^* : {}^7\text{Li} = \frac{1}{2}$.

It has been pointed out by HAFSTAD and TELLER [38] that the α-particle model offers another possibility of interpretation not subject to such a difficulty. According to this model, the states of lowest energy of the ^{7}Li nucleus are characterized by the wave-function of the loose proton "hole" (*13.22*). Let ψ_1, ψ_2 represent the wave-function of a nucleon in the field of the first or the second α-particle of the ^{8}Be system; the proton hole eigenfunction may then be either $\psi_1 - \psi_2$ or $\psi_1 + \psi_2$, with one or no node, respectively, at the centre of gravity. The energy difference between the two corresponding stationary states (of which the lower corresponds to the former eigenfunction) will be given by twice the exchange integral $\int \psi_1 \mathcal{H} \psi_2$ ($\mathcal{H}$ denoting the Hamiltonian of a nucleon in the field of the two α-particles). It is difficult to arrive at anything more than a very rough estimate of this quantity, but it is certainly not unreasonable to envisage the possibility of identifying the excited state of ^{7}Li found empirically with the nodeless state $\psi_1 + \psi_2$ predicted by the α-particle model. A variation of the relative yield of two such states of different symmetry with the excitation of the compound nucleus would not be unexpected. On the other hand, it is necessary for the validity of this interpretation that the

17.51. *Excited state of ^7Li*

Nuclear reaction	Energy of impinging particles (MeV)	Yield $^7Li^*/^7Li$	Excitation energy of $^7Li^*$: from proton or α-particle groups (MeV)	Excitation energy of $^7Li^*$: from γ-ray (MeV)	Reference
Compound nucleus ^8Be					
$^6Li(d,p)$	0,1...0,9	0,25...0,74	$0{,}455 \pm 0{,}015$	$0{,}400 \pm 0{,}025$	WILLIAMS *et al.* [37], RUMBAUGH *et al.* [38]
$^7Li(p,p)$	1,080; 1,290			$0{,}495 \pm 0{,}025$	FOWLER and LAURITSEN [39]*
	1,032; 1,214; 1,639			0,459	HUDSON, HERB and PLAIN [40]
$^6Li(d,n)^7Be \xrightarrow{K} {}^7Li$	1			$0{,}425 \pm 0{,}025$	RUMBAUGH *et al.* [38]
	2,7			$0{,}485 \pm 0{,}005$	ZLOTOWSKI and WILLIAMS [42]**
	6			$0{,}453 \pm 0{,}005$	SIEGBAHN [46]
	?			$0{,}465 \pm 0{,}010$	RUBIN [41]
Compound nucleus ^{11}C					
$^{10}B(p,\alpha)^7Be \xrightarrow{K} {}^7Li$	0,95			$0{,}425 \pm 0{,}025$	RUMBAUGH *et al* [38]
	0,85			$0{,}425 \pm 0{,}020$	MAIER-LEIBNITZ [38]
Compound nucleus ^{11}B					
$^{10}B(n,\alpha)$	slow	15	0,42		BØGGILD [45]***
$^9Be(d,\alpha)$	0,2...0,35	1,7...1,115	$0{,}494 \pm 0{,}016$		Miss GRAVES [40]***
$^7Li(\alpha,\alpha)$	5,29			0,462	SIEGBAHN and SLÄTIS [47] †

* According to HUDSON, HERB and PLAIN [40], this value should be lowered by applying a correction to the measured absorption coefficient of the γ-ray, and should then agree with theirs.

** According to SIEGBAHN [46], a lower value might be derived from the data.

*** These papers contain reviews of older work.

† This measurement disproves the occurrence of more than one γ-ray in this reaction, as claimed in an early work by Bothe.

splitting of either of the states $\psi_1 - \psi_2$, $\psi_1 + \psi_2$ due to the interaction between the spin of the "hole" and its orbital momentum be of a smaller order of magnitude than the observed energy difference of 0,45 MeV; and in view of the conclusion derived from the ^{5}He case (*17.43*), this might appear rather doubtful. For ^{7}Li, only discussions of the small $^2P_{\frac{3}{2}} - {}^2P_{\frac{1}{2}}$ splittings caused by the relativistic spin-orbit couplings or the axial dipole interactions * are available, and these are clearly insufficient to settle the issue.

17.53. *Relativistic doublet splitting of ^{7}Li.* Although the relativistic effects are probably of little significance for the actual state of affairs, their discussion offers some interest in affording an illustration of the considerable differences which may arise between the predictions of different models of nuclear structure. The effect of relativistic spin-orbit coupling has been extensively treated on the quasi-atomic model by BREIT and STEHN [38*c*], in a paper containing also important results of general validity for any type of spin-orbit coupling; the α-particle model is examined in a paper by KITTEL [42], who lays special stress on the importance of the exchange integrals in the expression for the average interaction energy (*11.11*). Without entering into such involved calculations, we may, like INGLIS [39*c*, *a*], easily compare the orders of magnitude of the Larmor and Thomas terms on the two models.

These models may be said to differ in two respects; in the first place, by a different subdivision into "clusters" of nucleons assumed in first approximation to move as a whole: on the α-particle model, we have an α-particle and a triton; on the quasi-atomic model, we may in a rough way envisage the motion of the p-shell proton with respect to the residual ^{6}He cluster. The second difference concerns the forces acting between the clusters: these are weaker on the α-particle model than in the other case. While these differences do not very much affect the Larmor splitting, they have a considerable influence on the Thomas term. If M' denotes the reduced mass of the clusters with respect to the centre of gravity of the total system, the Larmor ^2P splitting may in fact be written

$$[{}^2\varepsilon^{(1)}_{\frac{3}{2}} - {}^2\varepsilon^{(1)}_{\frac{1}{2}}]_{\text{Larmor}} \approx -\frac{3}{4}\frac{\hbar^2}{MM'}\, g_s\, \frac{2e^2}{R^3}, \tag{1}$$

R denoting the average distance of the clusters, for which we may take the nuclear radius $R \approx 2r_0 \approx d$. Since $M' = \frac{12}{7} M$ on the α-particle model and $\frac{6}{7} M$ on the quasi-atomic model, the Larmor splitting is of the same order of magnitude in both cases, viz. $\approx 10^{-2}$ MeV. As regards the Thomas splitting, we can also write down in the same form in both cases the contribution arising from the acceleration due to the orbital motion of the clusters. Denoting by ω the angular velocity, divided by c, of this orbital motion, we have, for a P state,

$$\hbar \,|\, 2 \approx M' R^2 \omega; \tag{2}$$

on the other hand, the corresponding Thomas precession velocity

$$\omega_T \approx \tfrac{1}{2}\, \omega^3 R^2, \tag{3}$$

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* Using the α-particle model of the ^{7}Li nucleus, INGLIS [39*c*], by a very crude estimate, concludes that the (second order) splitting due to the axial dipole interaction of meson theory will be less than 0,015 MeV.

whence for the ^2P splitting

$$[{}^2\varepsilon^{(1)}_{\frac{3}{2}} - {}^2\varepsilon^{(1)}_{\frac{1}{2}}]_{\text{Thomas}} \approx \frac{3}{\sqrt{2}} \frac{\hbar^4}{M'^3 R^4}. \tag{4}$$

On the α-particle model, we should take account of a further radial acceleration, due to the virtual processes of exchange of nucleons between the clusters, which contribute to their interaction; but, as shown by Inglis, the resulting correction to the Thomas precession (3) is rather small. Keeping, therefore, to formula (4), we see that the occurrence of the reduced mass M' to a large power has the effect of reducing the Thomas effect by a factor 8 on the α-particle model. The stronger nuclear field acting on the proton in the quasi-atomic picture would tend to lower the value of R, which again would work in the same direction, viz. to reduce the ratio of the Thomas splittings predicted by the two models. Since for $M' \approx M$ the ratio

$$\left|\frac{\text{Larmor splitting}}{\text{Thomas splitting}}\right| \approx \left(\frac{e^2}{\hbar}\right)^2 \frac{M}{m} \approx \frac{1}{10}, \tag{5}$$

we see that while the whole effect is dominated, on the quasi-atomic model, by the Thomas contribution, this is no longer true on the α-particle model; in this case, both the Thomas and the Larmor contributions become of the same order of magnitude, much smaller than in the former case, or the Larmor effect even becomes preponderant.

APPENDICES

APPENDIX I

ALLOWED β-TRANSITIONS

A1.0. The theory of β-radioactivity is only indirectly connected with the main subject of this book. We shall here only develop the treatment of allowed β-transitions on the basis of Wigner's theory of supermultiplets (*10.14*). For other aspects of the theory and a survey of the empirical material, we can refer the reader to the up-to-date and authoritative account of KONOPINSKI [43].

A1.1. Allowed transitions on Wigner's approximation

A1.11. *General expression of life-time.* If a light nucleus goes over by a $\beta^{\pm}$-transition or a K-capture process from a state Ψ_i into a state Ψ_f, its life-time t_{if} in the initial state Ψ_i will be given in first approximation by a formula of the type

$$\frac{\log 2}{t_{if}} = \frac{1}{\tau_0} \,|\, G_{if} \,|^2 \, I(W_{if}), \tag{1}$$

in which τ_0 is a characteristic time-constant, G_{if} the matrix-element

$$G_{if} = \int \Psi_i^* \, \mathcal{V}_\beta \, \Psi_f \tag{2}$$

of a certain operator $\mathcal{V}_\beta$ characterizing the interaction between the nucleons and the lepton field, and $I(W_{if})$ a function of the total energy W_{if} (in units m) emitted in the transition:

$$\begin{aligned} &\beta^{\pm}\text{-transition:} \qquad W_{if} = 1 + \frac{K_\beta}{m} \\ &K\text{-capture:} \qquad W_{if} = \frac{K_\nu}{m}; \end{aligned} \tag{3}$$

K_β is the maximum kinetic energy of the emitted electron, while K_ν is given (*1.131–5*) by the difference of the atomic weights of initial and final nuclei. The function $I(W)$, which also depends on the charge number Z of the decaying nucleus, is a complicated expression, which need not be given in full here; we shall only mention the approximate formulae valid for small Z:

$$\begin{aligned} &\beta^{\pm}\text{-transition:} \quad I(W) = \sqrt{W^2-1}\,(\tfrac{1}{30}\, W^4 - \tfrac{3}{20}\, W^2 - \tfrac{2}{15}) \\ &\qquad\qquad\qquad\qquad\qquad + \tfrac{1}{4}\, W \log (W + \sqrt{W^2-1}) \\ &K\text{-capture:} \qquad I(W) = 2\pi\, (Z_{\text{eff}}/137)^3\, W^2; \end{aligned} \tag{4}$$

Z_{eff} denotes the effective charge number of the nucleus for the K electrons (e.g. $Z_{\text{eff}} = 3{,}7$ for $_4\text{Be}$). When the matrix-element G_{if} does not vanish, one speaks of an *allowed* transition.

In this approximation, there are only two essentially different possibilities for the interaction operator $\mathcal{V}_\beta$, viz. the *Fermi interaction*

$$\mathcal{V}_\beta = \sum_i \tau_1^{(i)} \tag{5}$$

and the *Gamow-Teller interaction*

$$\mathcal{V}_\beta = \frac{1}{\sqrt{3}} \sum_i \tau_1^{(i)} \vec{\sigma}^{(i)}. \tag{6}$$

The integral sign in (2) implies summation over spin and isotopic variables as well as integration over space coordinates of the nucleons. In the case of Gamow-Teller interaction, $|G_{if}|^2 = \sum_x |G_x|^2$. The operators $\tau_1^{(1)}$, or more generally (*4.12*) $\tau_\perp^{(i)}$, occur in (5) and (6) as the sums of the operators Π and $\Pi^\dagger$ effecting the transition from a neutron to a proton state and vice versa. To each type of interaction belongs a specific time-constant τ_0; one might, of course, more generally take a linear combination of the operators (5) and (6); the right hand side of (1) would then be replaced by a sum of similar expressions *.

If there is only one possible transition $i \to f$, the resulting β-spectrum will be called *simple*. In general, we may expect that there will be several allowed final states, of different energies. The observed β-spectrum will in such cases be a superposition of simple spectra with different upper energy limits W_{if} and different half-lives t_{if}; the observed half-life t is then given by

$$\frac{1}{t} = \sum_f \frac{1}{t_{if}} = \frac{1}{\tau_0 \log 2} \sum_f |G_{if}|^2 I(W_{if}). \tag{7}$$

Even if the separate $|G_{if}|^2$ may be difficult of evaluation, their sum $\sum_f |G_{if}|^2$ over all possible final states has a simple meaning: since $\mathcal{V}_\beta$ is Hermitian, it is just the expectation value of $\mathcal{V}_\beta^2$ in the initial state.

A1.12. *Selection rules.* Thanks to the connexion of the operators (5) and (6) with infinitesimal rotations in ordinary or symbolic space, it is possible, as emphasized by WIGNER [39*b*], to discuss the matrix-element (2) to a large extent without any explicit knowledge of the wave-functions. We may first enunciate selection rules of universal validity, pertaining to the parity and total angular momentum of the levels between which the

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* Strictly speaking, the Gamow-Teller operator (6) itself arises from the linear combination of two distinct operators, viz. (6) and that obtained by substituting $\varrho_3^{(i)} \vec{\sigma}^{(i)}$ for $\vec{\sigma}^{(i)}$, which become identical in non-relativistic approximation for the nucleons. This gives rise, by interference in the calculation of the total $|G_{if}|^2$, to a factor of the form $\left(1 + \frac{\lambda}{W}\right)$, with $|\lambda| \leq 1$, multiplying the matrix-element $|G_{if}|^2$ defined by (2) and (6): we shall, however, disregard this correction. A linear combination of interactions (5) and (6), on the other hand, does not give rise to interference terms. See FIERZ [37].

transition takes place. It is clear that both levels must have *the same parity*. Further, since the interaction operators (5) and (6) behave as a scalar and a vector, respectively, with respect to rotations, the changes ΔJ, Δm of the quantum numbers J, m of the total angular momentum and its z-component are subject to the restrictions:

$$\text{Fermi interaction:} \qquad \Delta J = \Delta m = 0 \tag{8}$$

$$\text{Gamow-Teller interaction:} \qquad \Delta J \text{ and } \Delta m = 0 \text{ or } \pm 1. \tag{9}$$

Assuming now the validity of Wigner's approximation, we see that allowed transitions can only occur between states belonging to the same supermultiplet: *all transitions between states of different supermultiplets are forbidden.* In general, the initial state will be the ground state of the nucleus; its quantum number T_3 will therefore (*10.311*–18) be the largest in absolute value occurring in the supermultiplet. If $T_3 > 0$, any β^+-transition (or K-capture), $T_3 \rightarrow T_3 + 1$, will then be forbidden, since it would lead to another supermultiplet; but for $T_3 < 0$ β^+-transitions will be allowed. As regards β^--transitions, $T_3 \rightarrow T_3 - 1$, they will generally be impossible because, on account of the Coulomb interaction, the energies of the states of a supermultiplet increase with decreasing T_3; the chief* exception to this situation arises from the fact (*10.311*–22) that the ground states of the nuclei of mass number $4a + 2$ with $T_3 = 0$ as well as $T_3 = \pm 1$ both belong to the same supermultiplet (1,0,0) : if, then, the odd isobar ($T_3 = 0$) is more stable than the even one with $T_3 = 1$, the latter will transmute into the former with allowed emission of a negaton. If the odd isobar is unstable, it will undergo an allowed β^+-decay. We thus arrive at the general conclusion that *most β-transitions from light nuclei are forbidden; there are, however, two exceptional classes of β-active nuclei:*

(*a*) positon emitters with negative neutron excess,
(*b*) negaton or positon emitters of mass number $4a + 2$, the ground state of which belongs to the supermultiplet (1,0,0).

To class (*a*) belongs, in particular, the well-known series with $T_3 = -\frac{1}{2}$ (*3.3*); the corresponding supermultiplets are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, according as A is of the form $4a + 1$ or $4a + 3$ (*10.311*–19, 21).

In addition to the general selection rules (8), (9), we now get, on Wigner's approximation, the analogous ones for the spin quantum number:

$$\text{Fermi interaction:} \qquad \Delta S = 0 \tag{10}$$

$$\text{Gamow-Teller interaction:} \qquad \Delta S = 0 \text{ or } \pm 1. \tag{11}$$

Moreover, is follows from the scalar character of both types of interaction (5) and (6) with respect to spatial rotations (without rotation of spins)

* Another exceptional case of allowed β^--decay is that of ^3H; see *A 1.212*.

that *no change of orbital momentum* can take place:

$$\Delta L = 0. \tag{12}$$

Combining (8), (10) and (12), we see that *on Fermi interaction* there is at most one transition of definite (β^+ or β^-) type from any initial state: *the β-spectrum is simple*. Clearly, this conclusion remains strictly valid even when the spin dependent nuclear forces are not neglected and mix up different supermultiplets. But if the interaction is of the Gamow-Teller type, the β-spectrum will generally be complex and when Wigner's approximation is abandoned, no regular relation between life-time and energy can any more be expected. Since the number of possible transitions from a given initial state will become larger, while the sum $\sum_f |G_{if}|^2$ remains bounded, each $|G_{if}|^2$ will then on the average be much smaller than on Wigner's approximation.

A1.13. *Calculation of matrix-elements.* The interpretation of $\sum_f |G_{if}|^2$ as expectation value of V_β^2 makes its calculation possible for any state belonging to a supermultiplet of either of the above classes (*a*) and (*b*) giving rise to allowed transitions. Owing to the property of V_β^2, of representing an infinitesimal rotation operator, its expectation value will depend only on the rotation quantum numbers of the state *i*, but not on the number of particles of the nuclear system: we can therefore simplify its calculation by performing it for the smallest value of *A* for which the supermultiplet containing state *i* exists. In the case of Fermi interaction, this gives at once the only $|G_{if}|^2$ occurring. On Gamow-Teller interaction, the relative values of the various $|G_{if}|^2$ may be derived by the usual procedures of atomic spectroscopy; in this case, a complete determination of all $|G_{if}|^2$ can thus also be achieved.

A1.131. *Matrix-elements for class* (*a*) *nuclei of neutron excess −1.* As regards the nuclei of class (*a*), we shall confine the discussion to those of neutron excess −1, the ground state of which belongs to one of the supermultiplets $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$. An example of the supermultiplet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is furnished by a single nucleon. Then $\sum_f |G_{if}|^2 = 1$ for all states *i* of the supermultiplet. The same holds for the supermultiplet $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$; for we may, to discuss this case, consider a system of 3 nucleons held together by such forces as to allow of a description by a quasi-atomic model: if they occupy an *s* shell, any matrix-element $|G_{if}|$ can be different from zero only if the final state *f* differs from the initial one *i* by just one individual wave-function ψ_i being replaced by another, ψ_f, in which case

$$|G_{if}| = \left|\int \psi_i^* \tau_1 \psi_f\right| \quad \text{or} \quad \left|\int \psi_i^* \tau_1 \vec{\sigma}\, \psi_f\right|,$$

exactly as in the one-nucleon problem (we may say that the matrix-element refers to the "hole" in the closed *s* shell).

To proceed further in the case of Gamow-Teller interaction, we first observe that, since $S = \frac{1}{2}$, only transitions with $\Delta S = 0$ are possible. If $L = 0$, there is only one such transition; hence, *for any interaction, if the ground state of the decaying nucleus is a* $^{22}S_{\frac{1}{2}}$ *state, the* β^+*-spectrum is simple.* For other L-values, we get a compound spectrum; the $|G_{if}|^2$ are then given by expressions similar to the well-known Hönl-Kronig formulae, in which, however, the roles of S and L are interchanged:

Ground state of decaying nucleus	Transition $J \to J'$	$\lvert G_{if}\rvert^2$ (Gamow-Teller int.)
$^{22}(L)_{L+\frac{1}{2}}$	$L+\frac{1}{2} \nearrow L+\frac{1}{2}$	$\frac{2L+3}{3(2L+1)}$
	$L+\frac{1}{2} \searrow L-\frac{1}{2}$	$\frac{4L}{3(2L+1)}$
$^{22}(L)_{L-\frac{1}{2}}$	$L-\frac{1}{2} \nearrow L+\frac{1}{2}$	$\frac{4(L+1)}{3(2L+1)}$
	$L-\frac{1}{2} \searrow L-\frac{1}{2}$	$\frac{2(L-1)}{3(2L+1)}$

(13)

A1.132. *Matrix-elements for class (b) nuclei.* The discussion of case (*b*) proceeds on the same lines. The simplest example of the supermultiplet (1,0,0) is given by a system of two nucleons. The symmetry character of the spatial wave-functions is then (*10.14*–8) $A(1+1)$ or $S(2)$, i.e. the eigenfunctions are symmetrical in the space coordinates. Their dependence on spin and isotopic variables is therefore either $^1(\tau)\,^3(\sigma)$ or $^3(\tau)\,^1(\sigma)$. Writing

$$(\tau_1^{(1)} + \tau_1^{(2)})^2 = 2\,(1 + \tau_1^{(1)}\tau_1^{(2)}),$$
$$\tfrac{1}{3}\,(\tau_1^{(1)}\vec{\sigma}^{(1)} + \tau_1^{(2)}\vec{\sigma}^{(2)})^2 = 2\,(1 + \tfrac{1}{3}\,\tau_1^{(1)}\tau_1^{(2)}\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}), \tag{14}$$

we easily get for the sums $\sum\limits_f |G_{if}|^2$:

Initial state	$\sum_f \lvert G_{if}\rvert^2$ Fermi interaction	$\sum_f \lvert G_{if}\rvert^2$ Gamow-Teller interaction
$^1(\tau)_0\ \ ^3(\sigma)$	0	$\frac{4}{3}$
$^3(\tau)_{\pm 1}\ \ ^1(\sigma)$	2	2
$^3(\tau)_0\ \ ^1(\sigma)$	4	0

(15)

There are two types of transitions involving a change $\Delta T_3 = \pm 1$:

$$^1(\tau)_0\,^3(\sigma) \rightleftarrows {}^3(\tau)_{\pm 1}\,^1(\sigma) \tag{16}$$

$$^3(\tau)_0\,^1(\sigma) \rightleftarrows {}^3(\tau)_{\pm 1}\,^1(\sigma). \tag{17}$$

Transitions (16) from or to triplet states involve a change of spin: they

are therefore forbidden on Fermi interaction, allowed on Gamow-Teller interaction. For transitions (17) between singlet states (with the additional restriction $\Delta L = 0$), the situation is just reversed: as shown by (15), they are forbidden on Gamow-Teller interaction, allowed on Fermi interaction.

For the allowed transitions, the $|G_{if}|^2$ are readily found. On Fermi interaction, we have, for any transition (17),

$$|G_{if}|^2 = 2, \tag{18}$$

as directly indicated by (15). (The value $\sum_f |G_{if}|^2 = 4$ for the transitions from ${}^3(\tau)_0\,{}^1(\sigma)$ corresponds in fact to *two* possibilities, viz. ${}^3(\tau)_{\pm 1}\,{}^1(\sigma)$, with equal $|G_{if}|^2$.) For Gamow-Teller interaction, the result is as follows:

A1.132. *Allowed β-transitions of class (b) nuclei, on Gamow-Teller interaction*

Transition	$T_3 \to T_3'$	$S \to S'$	$J \to J'$	$\lvert G_{if}\rvert^2 (L \neq 0)$	$\lvert G_{if}\rvert^2 (L=0)$
${}^1(\tau)_0\,{}^3(\sigma) \to {}^3(\tau)_{\pm 1}\,{}^1(\sigma)$	$0 \to \pm 1$	$1 \to 0$	$L+1 \to L$	$\frac{2}{3}$	$\frac{2}{3}$
			$L \to L$	$\frac{2}{3}$	—
			$L-1 \to L$	$\frac{2}{3}$	—
${}^3(\tau)_{\pm 1}\,{}^1(\sigma) \to {}^1(\tau)_0\,{}^3(\sigma)$	$\pm 1 \to 0$	$0 \to 1$	$L \to L+1$	$\frac{2}{3} \cdot \frac{2L+3}{2L+1}$	2
			$L \to L$	$\frac{2}{3}$	—
			$L \to L-1$	$\frac{2}{3} \cdot \frac{2L-1}{2L+1}$	—

A1.2. Discussion of empirical data

A1.21. *Class (a) nuclei of neutron excess −1.* As already mentioned (*3.3*), class (*a*) nuclei of neutron excess −1 have been the object of a systematic investigation. The evidence concerning them is presented in table *A1.21*. Columns 4 and 5 contain the observed half-lives t and maximum kinetic energies K_β of the emitted positons. In column 6, the value of the latter quantity, as calculated by formula (*3.3*–1) together with (*2.1*–1,2), is given: the good agreement with the observed value is of some theoretical significance, as explained in *3.3*. Column 7 gives the W derived by (*A1.11*–3) from the observed K_β (except for $A = 21$, where this quantity is not known experimentally). Column 8 lists the quantity

$$\tau_0' = \frac{1}{\log 2} I(W)\, t, \tag{1}$$

expressed in 10^3 sec; according to (*A1.11*–1), this quantity would be identical with the time-constant τ_0, if the β-spectrum were simple and the matrix-element $|G_{if}|^2$ equal to unity. Finally, column 3 gives the observed quantum number J' of the total angular momentum of the *product* nucleus in its ground state: this is the only certain element we can use at present to speculate on the possible transitions contributing to the observed spectrum.

The first case in the table (${}^7_4\mathrm{Be} \to {}^7_3\mathrm{Li}$) is not one of β^+-emission, but

of K-capture; accordingly, the value listed under K_β is in reality K_γ, and the W and $I(W)$ must be calculated by the appropriate formulae (*A1.11*–3, 4). The next two positon emitters (^{11}C and ^{13}N) have recently been studied with great care by K. SIEGBAHN and his collaborators [44; 45*a*, *b*] and we use their results. All other data concerning t, K_β (and r_0') are taken from KONOPINSKI [43], while the J' are given according to M&F.

A1.21. *Positon emitters of neutron excess —1.*

A	Isobaric pair	J'	t	K_β observed	K_β calculated	W	τ_0'
			sec	MeV	MeV		10^3 sec
7	$_4Be \rightarrow {_3Li}$	$\frac{3}{2}$	$5 \cdot 10^6$	0,87	—	1,7	2,58
11	$_6C \rightarrow {_5B}$	—	1230	0,98	1,01	2,92	4,82
13	$_7N \rightarrow {_6C}$	$\frac{1}{2}$	608	1,24	1,29	3,43	7,34
15	$_8O \rightarrow {_7N}$	$\frac{1}{2}$	125	1,72	1,62	4,37	5,99
17	$_9F \rightarrow {_8O}$	—	64	1,93	1,95	4,78	4,82
19	$_{10}Ne \rightarrow {_9F}$	$\frac{1}{2}$	20,3	2,20	2,27	5,31	2,65
21	$_{11}Na \rightarrow {_{10}Ne}$	—	23	—	2,56	6,00	7,68
23	$_{12}Mg \rightarrow {_{11}Na}$	$\frac{3}{2}$	11,6	2,82	2,87	6,52	4,45
25	$_{13}Al \rightarrow {_{12}Mg}$	—	7	2,98	3,17	6,85	3,46
27	$_{14}Si \rightarrow {_{13}Al}$	$\frac{5}{2}$	4,9	3,54	3,42	7,93	5,11
29	$_{15}P \rightarrow {_{14}Si}$	—	4,6	3,63	3,68	8,16	5,47
31	$_{16}S \rightarrow {_{15}P}$	$\frac{1}{2}$	3,2	3,86	3,94	8,57	4,69
33	$_{17}Cl \rightarrow {_{16}S}$	$\frac{5}{2}$	2,4	4,13	4,21	9,14	4,92
35	$_{18}A \rightarrow {_{17}Cl}$	—	1,9	4,37	4,46	9,57	4,97
37	—	—	—	—	—	—	—
39	—	—	—	—	—	—	—
41	$_{21}Sc \rightarrow {_{20}Ca}$	—	0,9	4,94	5,18	10,68	3,57

The question as to the complexity of the β-spectra is of course of considerable theoretical interest, as the existence of a compound spectrum would be a very strong argument in favour of Gamow-Teller interaction. Unfortunately, direct evidence in this respect is very scarce and inconclusive. In contradiction to previous belief, it has recently been established in a convincing way by K. SIEGBAHN *et al.* [45*a*, *b*], and by COOK *et al.* [47], that the β-spectrum of ^{13}N is simple. The latter authors state that they have been unable to detect any γ-ray in the interval 0,135 ... 0,700 MeV; they conclude that either the interaction is of the Fermi type, or the energy separation of the P doublet expected as the ground state of ^{13}C is outside the above interval (smaller than 0,135 MeV or larger than 0,700 MeV). Apart from this reliable case, we have only some evidence, by no means so secure, on the complexity of the transition $^7_4Be \rightarrow {^7_3Li}$. The ^7Li nucleus has a well-known stationary state of about 0,45 MeV excitation, and it seems that about 1/10 of the nuclei resulting from the decay of ^7Be are formed in this excited state: this can be concluded from the intensity of γ-rays of the same energy which accompany the disintegration. It is

rather pointless, however, to try to analyse this process with the help of the formulae (*A1.131*–13), for it is very doubtful (*17.52*) whether the ground state and the low excited state of ^7Li can be described as the $^2P_{\frac{3}{2}}$ and $^2P_{\frac{1}{2}}$ components of a doublet.

Should we do so all the same, we should first have to make some assumption on the quantum numbers (either $^2P_{\frac{3}{2}}$ or $^2P_{\frac{1}{2}}$) of the initial state of ^7Be. The ground state of ^7Li is taken to be the $^2P_{\frac{3}{2}}$ component of the doublet, in conformity with the observed angular momentum. The $I(W_{if})$ and $|G_{if}|^2$ being then determined, we get the fraction of transitions to the excited state of ^7Li as the ratio of the products $|G_{if}|^2 I(W_{if})$. By means of (*A1.11*–7) we can also derive the value of τ_0 from the observed half-life. The following results are obtained:

Transitions assumed	$\|G_{if}\|^2$	W_{if}	$I(W_{if})$	% transitions to excited state	τ_0 10^3 sec
$P_{\frac{3}{2}} \nearrow P_{\frac{3}{2}}$	$\frac{5}{9}$	1,7	$3{,}57 \cdot 10^{-4}$	19	1,69
$P_{\frac{3}{2}} \searrow P_{\frac{1}{2}}$	$\frac{4}{9}$	0,82	$0{,}83 \cdot 10^{-4}$		
$P_{\frac{1}{2}} \nearrow P_{\frac{3}{2}}$	$\frac{8}{9}$	1,7	$3{,}57 \cdot 10^{-4}$	3	2,35
$P_{\frac{1}{2}} \searrow P_{\frac{1}{2}}$	$\frac{1}{9}$	0,82	$0{,}83 \cdot 10^{-4}$		

Although the lack of data about the nature of the states involved in the process prevents us from submitting the theory to a closer test, we can, following WIGNER * and KONOPINSKI [43], draw some general conclusions from the consideration of the constant τ_0' listed in the above table. If we had to do with a Fermi interaction, this constant would be identical with τ_0 and should thus have the same value throughout. Considering the poor accuracy of the determinations of most life-times and upper limit energies, and the sensitiveness of $I(W)$ to even small changes of W, the relatively small fluctuations of the τ_0' values might be taken as supporting the theory based on Fermi interaction, were it not that weighty arguments can be adduced in favour of a predominance of Gamow-Teller interaction **.

In the latter case, we have generally to expect complex β-spectra. Since the value of W given in the table is the largest of the W_{if} and $I(W)$ is an increasing function of its argument, it follows from the formula (*A1.11*–7) for the observed life-time and the definition (1) of τ_0' that

$$\tau_0' \geqslant \frac{\tau_0}{\sum_f |G_{if}|^2}, \tag{2}$$

the sign of equality corresponding to the special case of a simple spectrum.

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* See WIGNER [39*b*] and a discussion by WIGNER at the end of the paper of WHITE *et al.* [41].

** These arguments are summarized by KONOPINSKI [43].

For the class of nuclei under consideration, this general formula reduces to (*A1.131*)

$$\tau_0' \gtrsim \tau_0 . \tag{3}$$

In view of this relation, it is interesting to observe that the states involved in the transition $^{19}_{10}\mathrm{Ne} \rightarrow {}^{19}_{9}\mathrm{F}$, which is one of those yielding the smallest τ_0' values, are probably S states (*10.32*), while the angular momentum of the product nucleus is known to be $\frac{1}{2}$: this is therefore probably a $^2S_{\frac{1}{2}} \rightarrow {}^2S_{\frac{1}{2}}$ transition, which (*A1.131*) indeed gives a simple spectrum. This argument would point to a value of

$$\tau_0 \approx 3 \cdot 10^3 \text{ sec} . \tag{4}$$

The association of a simple spectrum with a larger τ_0' value in the case of ^{13}N would then imply a failure of Wigner's approximation for the states playing a part in this decay process. On the other hand, the case of ^{7}Be seems to afford an example of a compound transition with exceptionally small τ_0'; the above calculation shows that this is at any rate compatible with a value of τ_0 of the order of magnitude given by (4), provided the initial state is assumed to have the angular momentum $\frac{1}{2}$.

A1.211. *Life-time of the neutron.* Formally, the neutron belongs to the class of nuclei just discussed. Disregarding the kinetic energy of the proton after the process, we have

$$W = \frac{M_n - M_p}{m} = 2{,}47$$

(*1.22*–5) and the corresponding $I(W)$ is given by (*A1.11*–4). We have here to do with a simple transition, for which $|G_{if}|^2 = 1$; taking the value (4) of τ_0, we compute by means of (*A1.11*–1) a half-life of about 25 minutes.

A1.212. *The β-decay of ³H and the mass of the neutrino.* The process $^3\mathrm{H} \xrightarrow{\beta^-} {}^3\mathrm{He}$, in which the states of both nuclei involved likewise belong to the supermultiplet $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, is an exceptional example of allowed β^--transition, due to the fact that the increase in Coulomb energy in this case is just small enough to make it possible; the maximum kinetic energy of the electron emitted is in fact extremely small: according to the latest estimate (WATTS and WILLIAMS [46]), $K_\beta < 0{,}03\, m$. On the other hand, the life-time of the triton has a surprisingly low value of about* 12 years; much too low, in fact, as emphasized by KONOPINSKI [47], to fit in with the evidence just discussed on class (*a*) nuclei, and especially with the estimate (4) for the constant τ_0. Since for the process under consideration, in which S states are involved, one expects (*A1.131*) $|G_{if}|^2 = 1$, this constant should be directly given by the quantity τ_0' defined by formula

* NOVICK [47] gives 12,1 ± 0,5 years, GOLDBLATT *et al.* [47] 10,7 ± 2 years.

(1). Now, for very small values of $W—1$, the expression (*A1.11*–4) for the factor $I(W)$ occurring in (1) reduces to

$$I(W) \approx 0{,}216\,(W—1)^{\frac{7}{2}}.$$

The above data then yield $\tau_0' \lesssim 0{,}55 \cdot 10^3$ sec. In order to remove this discrepancy, one would have to assume a maximum energy K_β as large as 0,05 m; this figure is perhaps not entirely ruled out in view of the difficulty of the measurements.

Konopinski attempted to link the discrepancy to the question of the mass of the neutrino. In fact, the above expression for $I(W)$ is only valid, strictly speaking, for a vanishing neutrino mass, or at any rate for values of W–1 sufficiently large compared with the ratio μ of this mass to that of the electron. In this respect, the case of ^{3}H, in view of the exceptionally small value of W–1, might require a quite different treatment. It is also necessary, as pointed out by PRUETT [48], to take into account the influence of the nuclear Coulomb field on the electron and a relativistic correction arising from the finite neutrino mass when summing over the spins of the emitted leptons. Pruett's conclusion is that the value of μ needed to account for the experimental data is in any case larger than 0,05 and might be as high as $\approx 0{,}3$ (if $K_\beta \approx 0{,}02\,m$).

However, such a large value of μ would definitely conflict with the evidence about the mass of the neutrino that can be derived from energy balances of cycles of nuclear reactions, mainly of the types

$$ {}^A_Z\text{Nucl}\,(n,p)\,{}^{\ A}_{Z-1}\text{Nucl} \quad , \quad {}^{\ A}_{Z-1}\text{Nucl} \xrightarrow{\beta} {}^A_Z\text{Nucl}$$

$$ {}^A_Z\text{Nucl}\,(p,n)\,{}^{\ A}_{Z+1}\text{Nucl} \quad , \quad {}^{\ A}_{Z+1}\text{Nucl} \xrightarrow{\beta+} {}^A_Z\text{Nucl},$$

or from the study of the shape of the β-spectrum near its end-point. In a comprehensive and critical survey of the relevant reaction cycles, FRANK [47] obtains the estimate $\mu = 0{,}0631 \pm 0{,}0701$; this means, allowing for any possible systematic error, that the neutrino mass is certainly less than $\frac{1}{10}$ of the electron mass, and is probably zero. A recent investigation by HUGHES and EGGLER [48] leads to even more precise conclusions. From a cloud chamber study of the reactions

$$ {}^3\text{He}\,(n,p)\,{}^3\text{H} \qquad \text{and} \qquad {}^{14}\text{N}\,(n,p)\,{}^{14}\text{C},$$

combined with the K_β values of the ^{3}H and ^{14}C β-spectra, they derived values of the neutrino mass amounting to a few keV, with a total margin of error of about 25 keV. This result puts the upper limit of μ at about 0,05, while pointing to a probably much smaller, or even vanishing, value. Besides, it shows that the neutrinos involved in the two processes studied have the same mass within about 5 keV; this information is interesting, as we shall see presently, in view of the anomalous character of the ^{14}C decay (*A1.22*).

The other method of ascertaining the mass of the neutrino necessitates

for its application a very accurate determination of the shape of the β-spectrum right up to its end-point. Quite recently, it has been possible for the first time to derive from such measurements on the spectrum of ^{35}S (COOK, LANGER and PRICE [48], an upper limit for the neutrino mass $\mu < 0{,}01$.

A1.22. *Class (b) nuclei.* All known data on class (b) nuclei are collected in the following table (KONOPINSKI [43]*; the last entry according to M & F). The third column indicates the type of decay, as defined by the transition $T_3 \to T_3'$; the last one again gives the quantity τ_0' defined by (1).

A1.22. *Decay processes of nuclei of mass 4a+2, involving an odd isobar without neutron excess*

A	Isobaric pair	Type of decay ($T_3 \to T_3'$)	t sec	K_β MeV	W	τ_0' 10^4 sec
6	$_2He \to {}_3Li$	β^- $(1 \to 0)$	0,8	3,7	8,25	1,67
10	$_4Be \to {}_5B$	β $(1 \to 0)$	$9{,}1 \cdot 10^{13}$	0,56	2,1	$8{,}2 \cdot 10^{10}$
10	$_6C \to {}_5B$	β^+ $(-1 \to 0)$	9	3,35	7,57	8,6
14	$_6C \to {}_7N$	β^- $(1 \to 0)$	$1{,}5 \cdot 10^{11}$	0,143	1,28	$8{,}7 \cdot 10^5$
18	$_9F \to {}_8O$		6720	0,7	2,37	8,42
22	$_{11}Na \to {}_{10}Ne$		$>10^8$	0,55	2,08	$>2 \cdot 10^4$
26	$_{13}Al \to {}_{12}Mg$		7	2,98	6,85	3,46
30	$_{15}P \to {}_{14}Si$		153	2,99	6,87	72
34	$_{17}Cl \to {}_{16}S$	β^+ $(0 \to 1)$	1980	2,5	5,90	403
38	$_{19}K \to {}_{18}A$		462	2,3	5,50	58
42	$_{21}Sc \to {}_{20}Ca$		$1{,}2 \cdot 10^6$	1,4	3,74	$1{,}9 \cdot 10^4$
—	—		—	—	—	—
58	$_{29}Cu \to {}_{28}Ni$		81 or 474	—	—	—

Since (*11.41*) the ground state of the odd isobar is normally a triplet state, all the transitions listed in this table would be forbidden on Fermi interaction (*A1.132*). Now, the large values of τ_0' occurring in most cases are readily explained by supposing that the corresponding transition is a forbidden one. But the small τ_0' values obtained for

$$^{6}_{2}He, \quad ^{10}_{6}C, \quad ^{18}_{9}F, \quad ^{26}_{13}Al \tag{5}$$

cannot be accounted for in this way; on the contrary, their occurrence constitutes one of the main arguments for the preponderance of Gamow-Teller interaction.

On the quasi-atomic model (*10.24*), the ground states of the first three nuclei (5) would be S states, and this would also probably be the case

* More recent data: ^{10}Be HUGHES *et al.* [47], MCMILLAN [47]; ^{14}C REID *et al.* [46], LEVY [47] ($K_\beta = 0{,}154$ MeV), SALOMON, GOULD and ANFINSEN [47] (same K_β as Levy), NORRIS and INGHRAM [48] (half-life $1{,}6 \cdot 10^{11}$ sec).

for ${}^{26}_{13}\mathrm{Al}$. Table *A1.132* then shows that their β-decay would be simple; in the first two cases, one would have $\tau_0 = 2\tau_0'$, in the last two $\tau_0 = \frac{4}{3}\tau_0'$. The τ_0 values so obtained, viz.

$$3{,}34 \quad ; \quad 17{,}2 \quad ; \quad 5{,}61 \quad ; \quad 2{,}31 \tag{6}$$

(10^3 sec), are — except the second — of the required order of magnitude (4), determined by the evidence from class (*a*) nuclei. The larger τ_0' values appearing for higher mass-numbers (from $A = 30$ on) — and perhaps also the exceptionally high value for ${}^{10}_{6}\mathrm{C}$ — could be ascribed to the inadequacy of Wigner's approximation. In fact, for heavier nuclei, the perturbations due to the spin dependent nuclear forces would gradually tend to blur the distinction between allowed and forbidden transitions (*A1.12*).

Much more puzzling are the high τ_0' values corresponding to the transitions

$${}^{10}_{4}\mathrm{Be} \rightarrow {}^{10}_{5}\mathrm{B} \quad , \quad {}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} \quad , \quad {}^{22}_{11}\mathrm{Na} \rightarrow {}^{22}_{10}\mathrm{Ne}. \tag{7}$$

As compared with the allowed decay processes of the nuclei (5), these transitions are characterized by abnormally long life-times and abnormally low energy releases. It is not probable that their highly forbidden character can be ascribed to some anomalous behaviour of the initial or final state. Thus, it has been established, by the examination of the band spectrum, that the angular momentum of the nucleus ${}^{14}\mathrm{C}$ is zero (JENKINS [48]). As regards ${}^{10}\mathrm{Be}$, it would be difficult to understand how the initial state of this nucleus could be so very different from that of the other unstable isobar ${}^{10}\mathrm{C}$, which seems to behave normally: the two isobars differ only by the exchange of two neutrons for two protons, and (on the quasi-atomic picture) lie symmetrically with respect to ${}^{10}\mathrm{B}$, as regards the filling of the 2*p* shell (*10.32*); there can thus at most be a second order perturbation due to Coulomb interaction, and this effect, estimated by COOPER and NELSON [41], although working in the right direction, seems definitely too small. This deadlock has induced OPPENHEIMER [41] to put forth suggestions with far-reaching implications on the nature of the neutrinos involved in such anomalous processes; but this interesting speculation does not seem to be promising *.

The formally similar transition (*7.2*)

$${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{2}_{1}\mathrm{D} \tag{8}$$

requires some comment. As stated in *7.2*, one can use the cross-section for

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* Oppenheimer's suggestion that the neutrino in question would have a spin $\frac{3}{2}$ has been shown by KUSAKA [41] to lead to difficulties as regards the form of the spectrum. His assumption that there would be a difference in mass between the neutrinos taking part in "normal" and anomalous processes has been disproved by the work of Hughes and Eggler described in *A 1.212*.

the β^+-process (8), treated as an allowed transition, to investigate the rate of energy production in a white dwarf star as a result of this reaction. All published papers on this subject conclude that it is impossible on this basis to reconcile the observed low luminosities and relatively large masses of these stars. In other words, the calculated rate of the reaction (8) would seem to be much too fast. This conclusion would suggest that the process (8) behaves in the same anomalous way as the transitions (7), which would be equally hard to understand, since it is obvious that we have here a transition of the type $^1S \rightarrow {}^3S$. However, a renewed examination of the astronomical evidence has recently led Dr. SCHATZMAN to reverse the conclusion just mentioned. When due account is taken of the fact that the hydrogen in a white dwarf is localized in a thin superficial layer, there appears to be no difficulty in accounting for the rate of energy production by the reaction (8) considered as an allowed process.

APPENDIX II

ELECTROMAGNETIC PROPERTIES OF NUCLEI

A2.1. Multipole moments of a system of charged particles

A2.11. *System in slowly varying external field.* Let us consider a system of charged particles under the action of an external electromagnetic field. The distribution of electricity of the system is characterized by a current density $\vec{I}$ and a charge density ϱ, the external field by a vector potential $\vec{A}$ and a scalar potential B, from which the field strengths $\vec{E}$, $\vec{H}$ are derived in the usual way. The interaction energy, insofar as higher powers of the potentials may be neglected, has the form

$$V_{\text{el.m.}} = -\int \vec{A} \cdot \vec{I}\, dv + \int B \varrho\, dv, \tag{1}$$

where we must take for $\vec{I}$ and ϱ their expressions in absence of any electromagnetic field. More particularly, we shall consider a *slowly varying* field, i.e. assume that the relative variations of the field components over distances of the order of the dimensions of the system and times of the order of the proper periods connected with the system are small compared with unity. In such a case, it is well-known that expression (1) for the interaction energy can be put into a more convenient form, involving the values of the electromagnetic field and its derivatives at some arbitrarily chosen point O and the successive *multipole moments* of the system with respect to this point. The properties of the system to any desired approximation are then easily derived from the expressions for such multipole moments.

We shall here carry out the transformation of the operator (1) up to the second order of approximation, on the further assumption that the velocities of the particles are small compared with the velocity of light, i.e. that the dimensions of the system are small compared with the proper wave-lengths (MØLLER and ROSENFELD [43]). This approximation involves the few first multipole moments, defined by the following formulae, in which the vector $\vec{x}$ is taken from the point O as origin:

$$\begin{aligned}
&\text{the total charge} && e = \int \varrho\, dv \\
&\text{the electric dipole moment} && \vec{P} = \int \varrho \vec{x}\, dv \\
&\text{the magnetic dipole moment} && \vec{M} = \tfrac{1}{2} \int \vec{x} \wedge \vec{I}\, dv \\
&\text{the electric quadrupole moment}^{*} && Q_{ik} = \tfrac{1}{2} \int \varrho \left(x_i x_k - \tfrac{1}{3}\delta_{ik} r^2\right) dv.
\end{aligned} \tag{2}$$

* In the expression for Q_{ik}, the x_i $(i = 1, 2, 3)$ denote the Cartesian components of the vector $\vec{x}$.

We start from the expansions

$$\begin{aligned} B &= B_O + \vec{x}\,\mathrm{grad}_O B_O + \tfrac{1}{2}\,\vec{x}(\vec{x}\,\mathrm{grad}_O)\,\mathrm{grad}_O B_O + R \\ \vec{A} &= \vec{A}_O + (\vec{x}\,\mathrm{grad}_O)\vec{A}_O + \vec{R}', \end{aligned} \tag{3}$$

the residuals R, $\vec{R}'$ containing higher derivatives of the potentials with respect to the point O. Remembering further that

$$\int \vec{I}\,dv = \dot{\vec{P}} \tag{4}$$

and putting for a moment

$$Q'_{ik} = \tfrac{1}{2}\int \varrho\, x_i x_k\, dv\,,$$

we therefore get *

$$\begin{aligned} \int B\varrho\, dv &= eB_O + \vec{P}\,\mathrm{grad}_O B_O + (Q'\,\mathrm{grad}_O)\,\mathrm{grad}_O B_O + R'' \\ \int \vec{A}\vec{I}\, dv &= \dot{\vec{P}}\vec{A}_O + \int \vec{I}(\vec{x}\,\mathrm{grad}_O)\vec{A}_O\, dv + R''', \end{aligned} \tag{5}$$

the residuals being of higher order of approximation than the second. Now,

$$\begin{aligned} &\vec{I}(\vec{x}\,\mathrm{grad}_O)\vec{A}_O - \vec{x}(\vec{I}\,\mathrm{grad}_O)\vec{A}_O = (\vec{x}\wedge\vec{I})\vec{H}_O \\ &\vec{I}(\vec{x}\,\mathrm{grad}_O)\vec{A}_O + \vec{x}(\vec{I}\,\mathrm{grad}_O)\vec{A}_O = \vec{I}\,\mathrm{grad}\,[\vec{x}(\vec{x}\,\mathrm{grad}_O)\vec{A}_O]\,; \end{aligned}$$

whence, on account of $\mathrm{div}\,\vec{I} + \dot{\varrho} = 0$,

$$\int \vec{I}(\vec{x}\,\mathrm{grad}_O)\vec{A}_O\, dv = \vec{M}\vec{H}_O + (\dot{Q}'\,\mathrm{grad}_O)\vec{A}_O\,,$$

and

$$\begin{aligned} -\int \vec{A}\vec{I}\, dv &= \vec{P}\dot{\vec{A}}_O - \vec{M}\vec{H}_O + (Q'\,\mathrm{grad}_O)\dot{\vec{A}}_O \\ &\quad - \frac{d}{c\,dt}\,[\vec{P}\vec{A}_O + (Q'\,\mathrm{grad}_O)\vec{A}_O] - R'''. \end{aligned} \tag{5a}$$

The time derivative of an operator, occurring in this formula, can be omitted since it does not give any contribution either to the expectation value of the interaction energy in stationary states or to the matrix-elements corresponding to transitions between states of the system with the same total energy. From (5) and (5*a*), we thus finally get, apart from higher order residuals,

$$\mathcal{V}_{\mathrm{elm}} = eB_O - \vec{P}\vec{E}_O - \vec{M}\vec{H}_O - (Q\,\mathrm{grad}_O)\vec{E}_O - \ldots. \tag{6}$$

In this formula, it was possible to replace Q' by the traceless tensor Q, as defined by (2), because, for an external field, $\mathrm{div}_O\,\vec{E}_O = 0$. The various

* If Q is a tensor and $\vec{u}$ a vector, $Q\vec{u}$ represents the vector with components $\sum_k Q_{ik} u_k$.

terms of the expansion (6) have familiar interpretations, which it is superfluous to recall. We shall here especially be concerned with the quantities known in spectroscopy under the names of *magnetic moment* and *electric quadrupole moment:* these are not the vectors $\vec{M}$ or tensors Q themselves, but certain mean values of components of these quantities, which we shall now proceed to define.

A2.12. *Magnetic moment in spectroscopic sense.* In the first place, we deal, in the theory of hyperfine structure, with the expectation value of the electromagnetic energy in the ground state of the nucleus investigated. As is well-known, only the projection of $\vec{M}$ on the direction of the total angular momentum contributes to the expectation value of this vector in any state in which the total angular momentum is fixed. Calling this angular momentum $\hbar\vec{J}$ and taking the nuclear magneton (*1.21*–2) $\mu_0 = \hbar e/2M_p$ as unit, we may thus write

$$\mathrm{av}\,\vec{M} = \frac{\mu}{J}\,\mu_0\,\mathrm{av}\,\vec{J}, \tag{7}$$

μ being a constant which may depend on the quantum number J of the state considered, but not on the magnetic quantum number m defining the "orientation" of the vector $\vec{J}$. The convention usually adopted is just to call this constant μ the *magnetic moment:* it may be described as the expectation value (in units μ_0) of the projection M_z of $\vec{M}$ on some fixed direction in the substate of maximum magnetic quantum number ($m = J$).

A2.13. *Electric quadrupole moment in spectroscopic sense.* The definition of the electric quadrupole moment in spectroscopic sense rests on a similar basis (CASIMIR [36], § 4): the expectation value of Q can be split into two factors, one of which is independent of the magnetic quantum number, while the other is the expectation value of the tensor $J_iJ_k - \frac{1}{3}\,\delta_{ik}\,\vec{J}^2$. The first factor can therefore again be determined by considering one of the components of Q, Q_{33} say, and taking its expectation value in some suitable substate; the definition of the *electric quadrupole moment* conventionally given is the expectation value of $6 \cdot \frac{1}{e} \cdot Q_{33}$ in the substate of maximum magnetic quantum number, i.e.

$$Q = \mathrm{av}_{(m=J)}\,\frac{1}{e}\int \varrho\,(3z^2 - r^2)\,dv, \tag{8}$$

the coordinates being referred to the fixed point O as origin. The factor $1/e$ means that the spatial distribution of the particles rather than the electric density is considered, so that Q has the dimension of the square of a length. For the deuteron, in particular, we can say, more simply, that the quadrupole moment is the average value, in the substate with $m = J$, of the operator

$$Q = \tfrac{1}{4}(3z^2 - r^2), \tag{9}$$

expressed in *relative* coordinates. Since av $(J_i J_k - \frac{1}{3}\delta_{ik}\vec{J}^2)$ vanishes identically for $J = 0$ or $\frac{1}{2}$, an electric quadrupole interaction occurs only for nuclei with angular momentum $J \geqq 1$.

A2.14. *Higher multipole moments.* In the application of equation (6) to problems of hyperfine structure, the ratio of successive terms of electric or magnetic origin is of the order of magnitude of the ratio of nuclear to atomic dimensions. This explains why no higher than electric quadrupole moments of nuclei have ever been detected. The possible occurrence of a magnetic octupole moment in the case of Iodine has been discussed, with negative outcome, by CASIMIR and KARREMAN [42].

A2.2. Magnetic moments of nuclei

A2.21. *The empirical data.* Our knowledge of magnetic moments of nuclei enables us to enunciate some general regularities, which indirectly throw light on the properties of nuclear forces:

(1) *Even nuclei* have no magnetic moment. This is, of course, in agreement with the conclusion (*10.24*) that the ground states of such nuclei are ^1S configurations.

(2) The magnetic moments of stable *odd nuclei* (given in the accompanying table *A2.21*–1 according to MILLMAN and KUSCH [41]) exhibit a regular decrease, which, as we shall see, can be brought in relation with an increasing influence of non-central nuclear couplings (*15.1*).

A2.21–1. *Magnetic moments of stable odd nuclei*

^2_1D	0,857
^6_3Li	0,821
$^{10}_5B$	0,598
$^{14}_7N$	0,403

(3) From the list *A2.21*–*2* of the magnetic moments of *odd mass nuclei* hitherto measured, two remarkable regularities emerge, the bearing of which on problems of nuclear structure will also presently be discussed:

(*a*) when plotted against the total angular momentum (fig. *A2.21*–1, 2), the magnetic moments fall into two groups, roughly distributed along two separate curves; this division obtains separately for the odd proton and the odd neutron nuclei, the curves being quite different in the two cases;

(*b*) if two isotopes differing by two neutrons have the same angular momentum, they mostly have also the same magnetic moment. This last regularity is strikingly exhibited by the figures in the fourth column (of both left and right half) of table *A2.21*–*2*, which gives the ratios μ_{A+2}/μ_A of the magnetic moments of the isotopes of mass $A + 2$ and A.

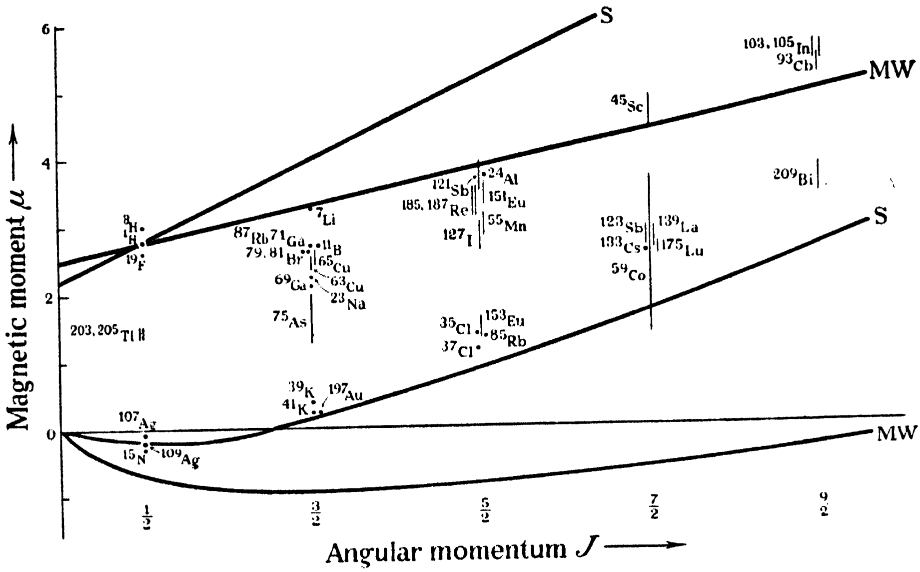

Fig. **A2.21-1.** Magnetic moments of *odd proton* nuclei. — Limiting curves (*A2.23*) according to Margenau and Wigner (*MW*) and to Schmidt (*S*).

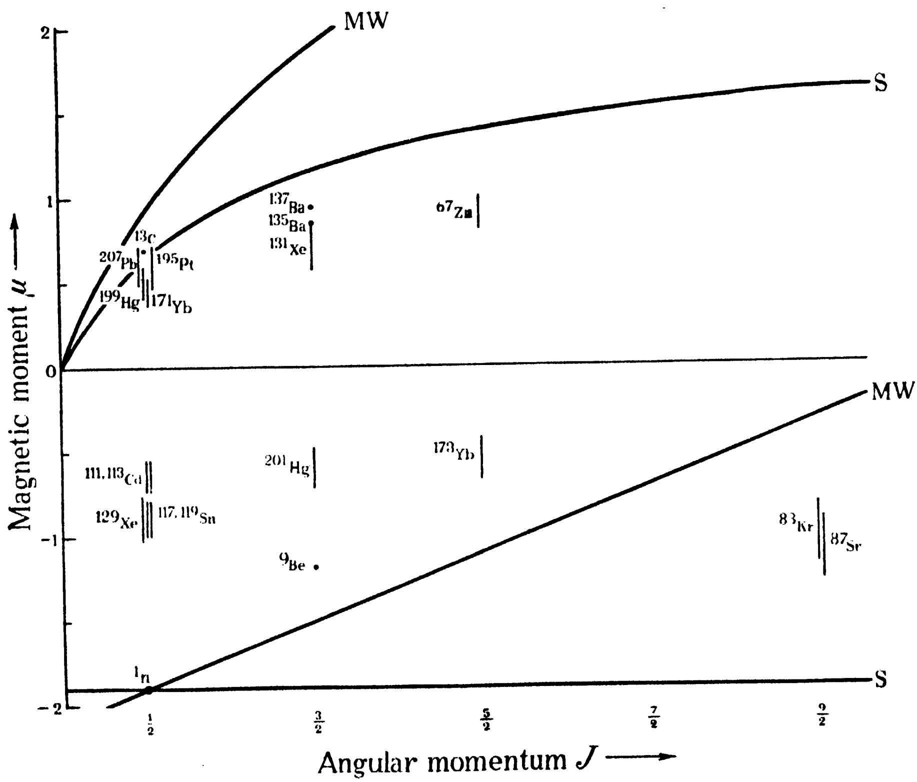

Fig. **A2.21-2.** Magnetic moments of *odd neutron* nuclei. — Limiting curves (*A2.23*) according to Margenau and Wigner (*MW*) and to Schmidt (*S*).

A2.21-2. *Magnetic moments of odd mass nuclei*

Odd proton nuclei				*Odd neutron nuclei*			
Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$	Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$
$^{1}_{1}H$	$\frac{1}{2}$	2,7928		n	$\frac{1}{2}$	−1,9125	
$^{3}_{1}H$	$\frac{1}{2}$	2,9791		$^{3}_{2}He$	$\frac{1}{2}$	(−) 2,131	
$^{7}_{3}Li$	$\frac{3}{2}$	3,2532					
				$^{9}_{4}Be$	$\frac{3}{2}$	−1,176	
$^{11}_{5}B$	$\frac{3}{2}$?	2,686					
				$^{13}_{6}C$	$\frac{1}{2}$	0,701	
$^{15}_{7}N$	$\frac{1}{2}$	(−)0,280					
$^{19}_{9}F$	$\frac{1}{2}$	2,625					
$^{23}_{11}Na$	$\frac{3}{2}$	2,217					
$^{27}_{13}Al$	$\frac{5}{2}$	3,630					
$^{35}_{17}Cl$	$\frac{5}{2}$	1,368	0,83				
$^{37}_{17}Cl$	$\frac{5}{2}$	1,136					
$^{39}_{19}K$	$\frac{3}{2}$	0,391	0,56				
$^{41}_{19}K$	$\frac{3}{2}$	0,215					
$^{45}_{21}Sc$	$\frac{7}{2}$	4,4...4,8					
$^{55}_{25}Mn$	$\frac{5}{2}$	3 0					
$^{59}_{27}Co$	$\frac{7}{2}$	2...3					
$^{63}_{29}Cu$	$\frac{3}{2}$	2,2265	1,07				
$^{65}_{29}Cu$	$\frac{3}{2}$	2,3847					
				$^{67}_{30}Zn$	$\frac{5}{2}$	0,9	
$^{69}_{31}Ga$	$\frac{3}{2}$	1,994	1,27				
$^{71}_{31}Ga$	$\frac{3}{2}$	2,540					
$^{75}_{33}As$	$\frac{3}{2}$	1,6					
$^{79}_{35}Br$	$\frac{3}{2}$	2,1066	1,08				
$^{81}_{35}Br$	$\frac{3}{2}$	2,2706					
				$^{83}_{36}Kr$	$\frac{9}{2}$?	− 1,0	
$^{85}_{37}Rb$	$\frac{5}{2}$	1,340					
$^{87}_{37}Rb$	$\frac{3}{2}$	2,733		$^{87}_{38}Sr$	$\frac{9}{2}$	≈ − 1,1	
$^{93}_{41}Cb$	$\frac{9}{2}$	5,3					
$^{107}_{47}Ag$	$\frac{1}{2}$	− 0,10	1,9				
$^{109}_{47}Ag$	$\frac{1}{2}$	− 0,19					

A2.21–2. *Magnetic moments of odd mass nuclei (Continued)*

Odd proton nuclei				*Odd neutron nuclei*			
Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$	Nucleus	J	μ	$\frac{\mu_{A+2}}{\mu_A}$
				$^{111}_{48}$Cd	$\frac{1}{2}$	−0,65	1,0
$^{113}_{49}$In	$\frac{9}{2}$	5,49	1,002	$^{113}_{48}$Cd	$\frac{1}{2}$	−0,65	
$^{115}_{49}$In	$\frac{9}{2}$	5,49					
				$^{117}_{50}$Sn	$\frac{1}{2}$	−0,89	1,0
				$^{119}_{50}$Sn	$\frac{1}{2}$	−0,89	
$^{121}_{51}$Sb	$\frac{5}{2}$	3,7					
$^{123}_{51}$Sb	$\frac{7}{2}$	2,8					
$^{127}_{53}$I	$\frac{5}{2}$	2,8					
				$^{129}_{54}$Xe	$\frac{1}{2}$	−0,9	
				$^{131}_{54}$Xe	$\frac{3}{2}$	0,7	
$^{133}_{55}$Cs	$\frac{7}{2}$	2,558					
				$^{135}_{56}$Ba	$\frac{3}{2}$	0,837	1,117
				$^{137}_{56}$Ba	$\frac{3}{2}$	0,936	
$^{139}_{57}$La	$\frac{7}{2}$	2,8					
$^{151}_{63}$Eu	$\frac{5}{2}$	3,4	0,441				
$^{153}_{63}$Eu	$\frac{5}{2}$	1,5					
				$^{171}_{70}$Yb	$\frac{1}{2}$	0,45	
				$^{173}_{70}$Yb	$\frac{5}{2}$	−0,65	
$^{175}_{71}$Lu	$\frac{7}{2}$	2,6					
$^{185}_{75}$Re	$\frac{5}{2}$	3,3	1,0108				
$^{187}_{75}$Re	$\frac{5}{2}$	3,3					
				$^{195}_{78}$Pt	$\frac{1}{2}$	0,6	
$^{197}_{79}$Au	$\frac{3}{2}$?	0.2					
				$^{199}_{80}$Hg	$\frac{1}{2}$	0,5	
				$^{201}_{80}$Hg	$\frac{3}{2}$	−0,6	
$^{203}_{81}$Tl	$\frac{1}{2}$	(1,612)	1,0097				
$^{205}_{81}$Tl	$\frac{1}{2}$	1.628					
				$^{207}_{82}$Pb	$\frac{1}{2}$	0,6	
$^{209}_{83}$Bi	$\frac{9}{2}$	3.6					

This table has been compiled from partial lists given by MARGENAU and WIGNER [40*b*], MILLMAN and KUSCH [41], and INGLIS [41]; full references to the sources will be found in these papers. Additional references: ^3H H. ANDERSON and A. NOVICK, *P.R.* **71**, 372. 1947; F. BLOCH, A. GRAVES, M. PACKARD and R. SPENCE, *P.R.* **71**, 373. 551. 1947; ^3He H. ANDERSON and A. NOVICK, *P.R.* **73**, 919. 1948; ^{13}C (spin) F. JENKINS, *P.R.* **72**, 169. 1947; ^{37}Cl (spin) E. SCHRADER, *P.R.* **64**, 57. 1943; $^{63,65}Cu$ R. POUND, *P.R.* **73**, 523. 1948; $^{69,71}Ga$ BECKER and KUSCH [48]; $^{79,81}Br$ S. BRODY, W. NIERENBERG and N. RAMSEY, *P.R.* **72**, 258. 1947; R. POUND, *P.R.* **72**, 1273. 1947; ^{93}Cb W. MEEKES and R. FISHER, *P.R.* **72**, 451. 1947; $^{113,115}In$ T. HARDY and S. MILLMAN, *P.R.* **61**, 459. 1942; ^{127}I (spin) W. GORDY *et al.* [47*a*, *b*]; ^{205}Tl H. POSS, *P.R.* **72**, 637. 1947.

Finally, we note that the spins and magnetic moments of two heavy odd nuclei of long life-time have been determined:

A2.21—3. *Magnetic moments of long-lived odd nuclei*

Nucleus	J	μ	Reference
$^{40}_{19}$K	4	$-1{,}290$	J. ZACHARIAS, *P.R.* **61**, 270. 1942
$^{176}_{71}$Lu	$\geqq 7$	3,8	(see INGLIS [41])

A2.22. *General expression for magnetic moment.* According to the terminology of *4.42*, the expectation value of the magnetic moment $\vec{M}$ consists of a direct contribution $\vec{M}_{\text{nucl}}$ of the constituent nucleons and an exchange moment $\vec{M}_{\text{exch}}$; the latter, however, may be expected to be small and is generally neglected *. The quantity $\vec{M}_{\text{nucl}}$ can further be analyzed into a term $\vec{M}_{\text{spin}}$ due to the spins of the nucleons and another $\vec{M}_{\text{orb}}$ due to the "orbital" motion of the protons:

$$\vec{M}_{\text{nucl}} = \vec{M}_{\text{spin}} + \vec{M}_{\text{orb}}, \tag{1}$$

with (*4.42*–9)

$$\vec{M}_{\text{spin}} = \sum_i \vec{\mu}^{(i)} = \mu_0 \sum_i (\mathrm{T}_+^{(i)} \mu_n + \mathrm{T}_-^{(i)} \mu_p)\, \vec{\sigma}^{(i)} \tag{2}$$

and

$$\vec{M}_{\text{orb}} = \mu_0 \sum_i \mathrm{T}_-^{(i)} \vec{l}^{(i)} \quad , \quad \vec{l}^{(i)} = \frac{1}{\hbar}\, \vec{x}^{(i)} \wedge \vec{p}^{(i)}. \tag{3}$$

The most comprehensive discussion of the expectation values of expression (1) is that given by MARGENAU and WIGNER [40*b*]. Before we proceed with this discussion, it will be good to recall a well-known ** theorem of quantum mechanics concerning average values of angular momenta, which is of constant application in calculations of magnetic moments. Consider two commuting vectors $\vec{A}$, $\vec{B}$, whose components satisfy the characteristic commutation relations of angular momenta, and let $\vec{C} = \vec{A} + \vec{B}$. Consider further a stationary state characterized by definite values of the lengths of the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, i.e. by quantum numbers A, B, C, such that $\text{av}\, \vec{A}^2 = A(A+1)$ etc. Then, the expectation value of the projection of $\vec{A}$ on some fixed direction Oz is given by

$$\text{av}\, A_z = \text{av}\, \frac{(\vec{A}\vec{C})}{C^2}\, C_z,$$

* For the exchange contribution to the magnetic moments of ^{3}H and ^{3}He, see *A2.251*.

** See, e.g., E. CONDON and G. SHORTLEY, *The Theory of atomic Spectra* (1935), Chapt. III, *10* (p. 64).

i.e., owing to the relation

$$2\vec{A}\vec{C} = \vec{C}^2 + \vec{A}^2 - \vec{B}^2, \tag{4}$$

by

$$\text{av}\, A_z = \frac{C(C+1) + A(A+1) - B(B+1)}{2C(C+1)}\, \text{av}\, C_z. \tag{5}$$

The most important contribution to the magnetic moment comes from the spins of the nucleons. With the notation (*10.14*–6, 7), the z-component of $\vec{M}_{\text{spin}}$ may be written

$$(M_z)_{\text{spin}} = \mu_0\, [(\mu_n + \mu_p) S_z - (\mu_p - \mu_n) \mathrm{Y}_z]. \tag{6}$$

The most convenient representation of the eigenfunction of the stationary state considered is therefore that based on the supermultiplets (P, P', P''), or more briefly P, of Wigner (*10.14*); each supermultiplet can further be analyzed into spin multiplets (*10.13*) with quantum numbers L, S, so that we have

$$\Psi = \Sigma c_{PLS}\, \Psi_{PLS}, \tag{7}$$

the sum involving all combinations compatible with the given value of the total angular momentum J; further, the eigenfunction Ψ refers to a definite magnetic quantum number m, which will ultimately be given the maximum value $m = J$ (*A2.12*). In the expectation value of the operator (6) in the state (7), non-diagonal elements of the type $\int \Psi^*_{PLS} \mathrm{Y}_z\, \Psi_{PLS'}$ will generally occur, but they may be neglected in comparison with the diagonal elements; if we denote the latter by the symbol av_{PLS}, we thus get

$$\text{av}\, (M_z)_{\text{spin}} = \Sigma\, |c_{PLS}|^2\, \text{av}_{PLS}\, (M_z)_{\text{spin}}\,; \tag{8}$$

but since $\text{av}_{PLS}\, \mathrm{Y}_z$ can be expressed, by application of formula (5), in terms of $\text{av}_{PLS}\, S_z$, we may define by

$$\text{av}_{PLS} (M_z)_{\text{spin}} = \mu_0\, g_s(PS)\, \text{av}_{PLS} S_z \tag{9}$$

a "Landé factor" or "gyromagnetic ratio" $g_s(PS)$ of the spin magnetic moment, which depends on the quantum numbers P and S, but not on L. Transforming the factor $\text{av}_{PLS} S_z$ in (9) by another application of (5), and inserting (9) in (8), we obtain finally for the spin contribution μ_s to the magnetic moment μ, defined in *A2.12*,

$$\mu_s = \Sigma\, |c_{PLS}|^2\, g_s(PS)\, \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}\, J. \tag{10}$$

The Landé factor g_s takes on a very simple form for the *lowest* supermultiplet $(PP'P'')$ of the nucleus under consideration. For odd mass nuclei, this supermultiplet, according to *10.311* is $(\frac{1}{2}n, \frac{1}{2}, \pm\frac{1}{2})$, the value $\mathrm{Y}_z = +\frac{1}{2}$ corresponding to an odd neutron, $\mathrm{Y}_z = -\frac{1}{2}$ to an odd proton; therefore,

$$\begin{aligned} g_s &= 2\mu_n \qquad \text{(odd neutron nuclei)} \\ g_s &= 2\mu_p \qquad \text{(odd proton nuclei).} \end{aligned} \tag{11}$$

The lowest supermultiplet of an odd nucleus corresponds in any case to a maximum spin quantum number $S = 1$, together with $Y_z = 0$; this gives

$$g_s = \mu_n + \mu_p = \mu_d^0 \qquad \text{(odd nuclei)}. \tag{12}$$

These results are, of course, obvious consequences of the fact that the lowest configurations of the nuclei in question are spin-saturated, while the odd proton and the odd neutron in an odd nucleus have parallel spins. In many calculations of magnetic moments it may be assumed that only the lowest supermultiplet has to be taken into account; this simplifies formula (10):

$$\mu_s = J g_s \left[\frac{1}{2} + \Sigma |c_{LS}|^2 \frac{S(S+1) - L(L+1)}{2J(J+1)}\right], \tag{13}$$

and it may further be noted that in this configuration no cross-terms such as have been neglected in formula (10) occur. The expression (13) is independent of any model of nuclear structure: if the quasi-atomic model is adopted, S and L have definite values, so that there is only one term in the sum. Odd mass nuclei with neutron excess ± 1 have also been treated on the α-particle model: the role of L is then played by the quantum number K (*13.22*), which may have the values $J \pm \frac{1}{2}$.

The calculation of the orbital magnetic moment does not lead to such results of fairly general validity: the use of different nuclear models here gives rise to rather divergent estimates. The eigenfunctions Ψ_{PLS}, or, if we drop for a moment the index of the supermultiplet, Ψ_{LS}, can be decomposed according to the eigenvalues of the total orbital momenta $\vec{L}_p$, $\vec{L}_n$ of the protons and the neutrons:

$$\Psi_{LS} = \sum_{L_p L_n} b_{LSL_pL_n} \Psi_{LSL_pL_n}. \tag{14}$$

Again, there occur in the expectation value $\mathrm{av}(M_z)_{\mathrm{orb}}$, besides the diagonal elements $\mathrm{av}_{LS}(M_z)_{\mathrm{orb}}$, non-diagonal terms

$$\int \Psi^*_{LS}(M_z)_{\mathrm{orb}} \Psi_{L\pm1,S}$$

which can be shown to be negligible. We thus have

$$\mathrm{av}\,(M_z)_{\mathrm{orb}} = \Sigma |c_{LS}|^2 \,\mathrm{av}_{LS}(M_z)_{\mathrm{orb}}, \tag{15}$$

and, using (5),

$$\mathrm{av}_{LS}(M_z)_{\mathrm{orb}} = \mu_0 g_l(LS)\, \mathrm{av}_{LS} L_z \tag{16}$$

with

$$g_l(LS) = \Sigma |b_{LSL_pL_n}|^2 \frac{L(L+1) + L_p(L_p+1) - L_n(L_n+1)}{2L(L+1)}. \tag{17}$$

Inserting (16) in (15) after a repeated application of (5), we get the orbital contribution μ_l to the magnetic moment in the form

$$\mu_l = \Sigma |c_{LS}|^2 g_l(LS) \frac{J(J+1) + L(L+1) - S(S+1)}{2J(J+1)} J. \tag{18}$$

Combining this with formula (10), we obtain for the magnetic moment μ

$$\mu = \mu_s + \mu_l = J\,\Sigma\,|c_{PLS}|^2 \Big[\tfrac{1}{2}[g_s(PS)+g_l(PLS)] + \tfrac{1}{2}[g_s(PS)-g_l(PLS)]\,\frac{S(S+1)-L(L+1)}{J(J+1)}\Big]. \tag{19}$$

The limitations of this general expression consist in the neglect of non-diagonal elements of the operators Y_z and $(M_z)_{\text{orb}}$.

On the *quasi-atomic model*, the ground state has definite quantum numbers L and S, and the sum on the right-hand side of (19) reduces to a single term. Moreover, if we assume that the multiplet to which the ground state belongs is either normal or inverted, this term takes a very simple form. In the case of an *inverted multiplet*, $J = L + S$ and

$$\mu = g_s S + g_l L\,; \tag{20}$$

for a *normal multiplet*, $J = |L - S|$ and

$$\begin{aligned} \mu &= \frac{J}{J+1}\,[(L+1)g_l - S g_s] \qquad (L > S) \\ \mu &= \frac{J}{J+1}\,[L g_l - (S+1) g_s] \qquad (L < S). \end{aligned} \tag{21}$$

On this model, the different configurations $\Psi_{LSL_pL_n}$ entering into the expansion (14) can, for not too heavy nuclei, easily be determined and the expansion coefficients b explicitly calculated in terms of the assumed central interaction potential. The Landé factor $g_l(LS)$ is then given by (17). Such calculations will be discussed in the sequel (*A2.252*).

For heavier nuclei, the best approach (MARGENAU and WIGNER [40*b*]) is provided by the *liquid droplet model*, according to which all constituent nucleons contribute equally to the expectation value of the orbital momentum *. This would mean

$$g_l(LS) \approx Z/A\,; \tag{22}$$

but formula (18) with this value of $g_l(LS)$ can give no more than a rough estimate of the orbital contribution μ_l, from which individual nuclei will more or less deviate. The treatment of light odd mass nuclei with neutron excess ± 1 on the *α-particle model* illustrates the kind of deviations that can be expected (SACHS [39]). In this case, the orbital momentum around the figure axis can be ascribed partly to the rotation of the nucleus as a whole, partly to an orbital motion around this axis of the extra particle or hole. To the first type of rotation a gyromagnetic ratio of the form (22) may be attributed, whereas the gyromagnetic ratio g_p of orbital motion is 1 if the additional or lacking particle is a proton and 0 if it is a neutron.

* Miss WAY [39*a*], in her treatment of the droplet model, makes the same assumption concerning the spin momentum. This, however, as emphasized by MARGENAU and WIGNER [40*b*], is hardly acceptable.

Consequently, the gyromagnetic ratio g_ζ pertaining to the component of orbital momentum around the figure axis will be of the form

$$g_\zeta = \theta g_p + (1-\theta)\frac{Z}{A} \qquad (0 \leqslant \theta \leqslant 1), \tag{23}$$

the parameter θ denoting the ratio of the contributions from orbital motion and bulk rotation to the total orbital momentum around the figure axis. The gyromagnetic ratio g_ξ corresponding to any component of orbital moment perpendicular to the figure axis will be just $\approx Z/A$, except when the configuration of the next α-nucleus is spherically symmetrical (e.g. ^{15}N, ^{15}O, see *13.21*), in which case it will be equal to g_ζ. The orbital magnetic moment arising from the bulk rotation is of the form (18), with $S = \frac{1}{2}$ and L (or rather K) $= J \pm \frac{1}{2}$, and the gyromagnetic ratio g_ξ. If g_ζ is different from g_ξ, it must be supplemented by a contribution μ_Λ from the orbital motion of angular momentum Λ (*13.21, 13.22*) around the figure axis, with gyromagnetic ratio $g_\zeta - g_\xi$. In order to calculate this contribution, we expand the eigenfunction of the ground state in terms of the eigenfunctions $\Psi_{Km_km_s}$ corresponding to a definite orbital momentum K and projection m_k of this orbital momentum on a fixed direction of space:

$$\Psi = \Sigma\, c_{Km_km_s}\, \Psi_{Km_km_s}; \tag{24}$$

since the projection m_s of the spin on the same direction is $\pm\frac{1}{2}$, while that of the total angular momentum, m, is specified (and taken $= J$), we have to take the summation over $m_k = m - m_s$ for $m_s = \pm\frac{1}{2}$, and over $K = J \pm \frac{1}{2}$. Calling Θ the angle between the fixed direction and the figure axis, we have

$$\mu_\Lambda = (g_\zeta - g_\xi)\,\Lambda \sum_{K,K'} \sum_{m_k,m_s} c^*_{Km_km_s} c_{K'm_km_s} (K|\cos\Theta|K')_{m_k\Lambda}; \tag{25}$$

in this formula, $(K|\cos\Theta|K')_{m_k\Lambda}$ represents the matrix-element of $\cos\Theta$ for specified values of the quantum numbers m_k, Λ, with respect to which it is diagonal *. The expansion coefficients c_{KS} or $c_{Km_km_s}$, indicating the proportions in which the two K states are mixed in the ground state, depend on the spin-orbit couplings (*A2.252*).

A2.221. *Conjugate and self-conjugate nuclei.* Owing to the symmetry of the nuclear interaction with respect to protons and neutrons (*3.3*), *light isobaric nuclei* (for which the asymmetry due to the Coulomb force can in first approximation be neglected) may be grouped in pairs of *conjugate* nuclei, with equal and opposite neutron excesses, having the same stationary states in that approximation: in two such conjugate nuclei, the respective

* These matrix-elements have been derived by H. CASIMIR, *Rotation of a rigid body in quantum mechanics* (thesis Leiden 1931), p. 50. In our notation,

$$(K|\cos\Theta|K) = m_k\Lambda / K(K+1)$$

$$(K|\cos\Theta|K-1) = (K-1|\cos\Theta|K) = \sqrt{K^2-m_k^2}\,\sqrt{K^2-\Lambda^2}/K\sqrt{4K^2-1}.$$

roles of protons and neutrons are simply interchanged. This implies a very simple and general relation, pointed out by SACHS [46a], between the magnetic moments μ^+ and μ^- of two conjugate nuclei. In fact, we get from (2) and (3)

$$\begin{aligned} \vec{M}^+_{\text{spin}} + \vec{M}^-_{\text{spin}} &= 2\mu_0(\mu_n+\mu_p)\vec{S} \\ \vec{M}^+_{\text{orb}} + \vec{M}^-_{\text{orb}} &= \vec{L}, \end{aligned} \tag{26}$$

whence, quite rigorously,

$$\begin{aligned} \mu^+ + \mu^- = \Sigma|c_{PLS}|^2 \Big[\mu_d^0 \frac{J(J+1)+S(S+1)-L(L+1)}{J(J+1)} \\ + \frac{J(J+1)+L(L+1)-S(S+1)}{2J(J+1)}\Big] J \\ = (\mu_d^0+\tfrac{1}{2})J + (\mu_d^0-\tfrac{1}{2})\,\Sigma|c_{PLS}|^2 \frac{S(S+1)-L(L+1)}{J(J+1)} J. \end{aligned} \tag{27}$$

This formula cannot, however, be checked by experiment for the time being, since at least one of the partners in any pair of conjugate nuclei is unstable and the technique of measuring magnetic moments of unstable nuclei has not yet been developed. The only exception concerns the pair of conjugate nuclei ^{3}H–^{3}He (*A2.251*).

An interesting case is that of the nuclei of zero neutron excess, which are *self-conjugate:* for those nuclei, the right-hand side of (27) represents twice the magnetic moment. In particular, the *stable odd nuclei,* for which (*11.41*) $J = 1$, have a magnetic moment of the form

$$\mu = \tfrac{1}{2}(\mu_d^0+\tfrac{1}{2}) + \tfrac{1}{2}(\mu_d^0-\tfrac{1}{2})\,\Sigma|c_{PLS}|^2 \frac{S(S+1)-L(L+1)}{2}: \tag{28}$$

this is the basic formula for the discussion of the empirical results (*A2.26*).

It may be observed that the orbital contribution to (27) and (28) can be deduced directly from (17), (18) by noticing that on account of the charge symmetry of nuclear interactions we have (provided the Coulomb force be neglected)

$$|b_{LSL_pL_n}|^2 = |b_{LSL_nL_p}|^2, \tag{29}$$

whence

$$g_l^+ + g_l^- = 1 \tag{30}$$

and for zero neutron excess

$$g_l = \tfrac{1}{2}. \tag{31}$$

A2.23. *The distribution of the magnetic moments of odd mass nuclei.* We are now in a position to discuss the regularities enumerated in *A2.21* from the theoretical point of view. We shall begin with the first collective property (3*a*) of odd mass nuclei, viz. the distribution of their magnetic moments along two curves, when plotted against the total angular momentum. If we assume that the ground states of the nuclei in question

belong to the lowest supermultiplet, their spin quantum number will be $S = \frac{1}{2}$ and they will in general consist of mixtures of two states with $L = J \pm \frac{1}{2}$. From this point of view, developed by MARGENAU and WIGNER [40*b*], we may draw two *limiting* curves, corresponding to the *pure* states with $L = +\frac{1}{2}$ and $L = J - \frac{1}{2}$: so far as our assumption as to the unperturbed lowest supermultiplet is valid, the magnetic moments should fall between these limiting curves. According to (20), (21), and (11), the limiting values of the magnetic moments are

$$\begin{aligned} L=J-\tfrac{1}{2}: &\qquad \mu = \mu_{pn} + (J-\tfrac{1}{2})\, g_l \\ L=J+\tfrac{1}{2}: &\qquad \mu = -\frac{J}{J+1}\,\mu_{pn} + \frac{J}{J+1}\,(J+\tfrac{3}{2})\, g_l\,, \end{aligned} \tag{32}$$

the symbol μ_{pn} denoting either μ_p or μ_n according as we are dealing with an odd proton or an odd neutron nucleus. Strictly speaking, these limiting moments are not merely functions of J, since g_l, even with its rough value (22), may be different for two nuclei with the same J; however, for our purpose of getting a general survey, we may adopt some constant value,

$$g_l \approx 0{,}4\,, \tag{32a}$$

say. In this way, we obtain the curves marked MW in the fig. *A2.21*–1, 2.

It must be admitted that the result is not very satisfactory. Not only does the theory fail to account for the separation of the empirical values into two groups within the above limits, but these limits themselves do not follow the general trend of the empirical curves. A much better agreement in this last respect is reached if, following the original suggestion of SCHMIDT [37], one adopts for g_l, instead of the value (32*a*) pertaining to the droplet model, that resulting from the rather extreme assumption that the orbital momentum is entirely due to the motion of the odd particle. This would mean that in the representation (14) of the eigenfunction, only one term would remain, for which L would be equal to L_p or L_n; the expression (17) for g_l then reduces to the standard value g_p for orbital motion (1 for a proton, 0 for a neutron):

$$g_l = g_p\,. \tag{33}$$

This value, inserted in (32), yields the curves marked S in fig. *A2.21*–1, 2. Curves closely representing the empirical separation into two groups would, as noticed by Schmidt himself and, more precisely, by INGLIS [38*a*], correspond to a sort of compromise between the extremes (32*a*) and (33), rather near to the latter, viz.

$$\begin{aligned} g_l &= \tfrac{1}{8} \qquad \text{(odd neutron nuclei)} \\ g_l &= \tfrac{7}{8} \qquad \text{(odd proton nuclei)}. \end{aligned} \tag{34}$$

The outcome of this analysis is therefore the disclosure of a strong influence of the *structure* of the nucleus on its angular momentum properties: it is rather surprising that this influence should go towards

preserving Russell-Saunders coupling and, above all, concentrating almost the whole orbital motion on a single nucleon.

The implications of this situation for the choice of the model of nuclear structure best adapted to the facts have been especially discussed in two papers by INGLIS [38a, 41]. He concludes that in this respect the α-particle model is superior to the quasi-atomic one. In the crudest approximation, both models concur in predicting that all the paired nucleons would tend to form a ^1S configuration, leaving only the orbital momentum (and spin) of the odd particle. But the question is in how far the perturbations due to the close coupling of the nuclear constituents will modify this simple picture. On the quasi-atomic model, one would expect such perturbations to be very severe, especially for heavier nuclei. On the α-particle model, on the other hand, it would seem that the ^1S configuration formed by the paired extra neutrons would be well isolated below the excited states, so that the odd proton would presumably be only weakly coupled with these neutrons. As regards the rotation of the framework of α-particles, its coupling with the odd proton would also be small on account of the exclusion of the low rotational levels *(13.15)*. Altogether, *a gyromagnetic ratio of the order (34) would* thus *seem plausible from the point of view of the α-particle model.*

A2.24. *Isotopes with equal magnetic moments.* We now turn to the other remarkable feature (*3b*) of odd mass nuclei mentioned in *A2.21*, viz. the existence of nuclei of this type, differing by two neutrons and having the same angular momentum and the same, or nearly the same, magnetic moment. From the consideration of this law, INGLIS [41] draws another argument in favour of the α-particle model; we shall summarize his discussion. At first sight, the approximate equality of magnetic moments seems to be simply accounted for by the droplet value $g_l \approx Z/A$ of the gyromagnetic ratio; but we have just seen that this value cannot be relied upon as describing the general behaviour of nuclear systems. We must therefore again look for an explanation in the structural properties of such systems. On the quasi-atomic model, one would expect in most cases the two neutrons to be added to an unfilled shell and to change g_l considerably. On the α-particle model, the degeneracy of the individual extra-neutron states would not be so large: in contrast to the quasi-atomic case, different orientations of the individual orbital momenta $\vec{l}^{(i)}$ relative to the body axes would involve different energies because of the lack of spherical symmetry. The two neutrons would thus often come to occupy an individual level which was previously empty; this would affect g_l only in higher order.

It may of course happen that — whatever the model of nuclear structure adopted — the addition of the two neutrons entails a large perturbation of the stationary state of the first isotope and a corresponding difference between the magnetic moments of the two nuclei. A concrete example illustrating this case is that of the pair of isotopes ^{39}K, ^{41}K, treated by

INGLIS [38*a*]. It will be convenient to use the description of the states concerned based on the quasi-atomic model. In conformity with the conclusions of *10.32*, the ground state of ^{39}K is represented by a configuration in which the 3*d* shell is entirely filled but for one proton: owing to this "hole", the protons form a ^2D configuration, while the neutrons are in a ^1S configuration; the resulting state of the nucleus is a $^2D_{\frac{3}{2}}$. Now, the two additional neutrons will be paired in the lowest level of the 3*p* shell, giving rise to a mixture of two states of the neutron system, viz. ^1S and ^1D. The state of the ^{41}K nucleus is thus still a $^2D_{\frac{3}{2}}$, but the admixture of ^1D neutron state will tend to lower the gyromagnetic ratio g_l, in agreement with experiment.

A2.25. *Magnetic moments of light odd mass nuclei.* The following considerations will now be concerned with more individual features, chiefly pertaining to the lighter nuclei, which are susceptible of detailed study. As regards odd mass nuclei, the comparison of the theoretical results with the observed values will throw some more light on the ability of the quasi-atomic and the α-particle model to account for finer features of nuclear magnetic moments.

A2.251. *The conjugate nuclei* ^3H *and* ^3He. The theoretical treatment of the simplest odd mass nuclei, ^{3}H and ^{3}He, is interesting in that it can be carried out completely and affords an example of perturbation of the lowest supermultiplet by higher ones *. The comparison between theory and experiment, on the other hand, has important implications with regard to fundamental aspects of the problem of nuclear forces.

In first approximation, the ground state of the nuclei in question is a $^2S_{\frac{1}{2}}$ one, antisymmetrical in the spins of the like particles (*14.111*), i.e. involving the spin eigenfunctions $^2\sigma_{\pm\frac{1}{2}}(\widetilde{12},3)$ (*14.111*–14), in which 1,2 refer to the like particles; this state, which belongs to the supermultiplet $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, will be more precisely denoted by $^2\widetilde{S}_{\frac{1}{2}}$. The magnetic moment is, in this approximation, μ_{pn}. However, the non-central nuclear interactions bring about (*17.2*) a mixture of this state with others, the effect of which on the magnetic moment we have to investigate. The admixed states consist of a $^2\widetilde{P}$ involving the same spin functions and belonging to the same supermultiplet, and of other states belonging to the supermultiplet $(\frac{3}{2},\frac{3}{2},\frac{3}{2})$, viz. a $^2\overline{S}$ and a ^2P involving the symmetric spin functions $^2\sigma_{\pm\frac{1}{2}}(12,3)$ (*14.11*–4), and two quartet states, ^4P, ^4D, with the spin functions $^4\sigma(\overline{12},3)$ (*14.11*–5), which are symmetrical in all three particles. We have to calculate the magnetic moments of all these states, and further to ascertain what combinations may give rise to non-diagonal terms in the expression for the magnetic moment of the ground state.

The spin gyromagnetic ratio g_s in the states $^2\widetilde{S}$, $^2\widetilde{P}$ is (as already stated) $2\mu_{pn}$. In the entirely symmetrical quartet states, we have, of course,

* The following treatment is an extension of that given by SACHS and SCHWINGER [46*b*].

av $\sigma_z^{(i)} = \frac{1}{3}$, and consequently $g_s = \frac{2}{3}(2\mu_n + \mu_p)$ for ^{3}H and $g_s = \frac{2}{3}(2\mu_p + \mu_n)$ for ^{3}He. Finally, the spin gyromagnetic ratio in the states $^2\overline{S}$, $^2\overline{P}$ is found, from the definition (9), by direct use of the explicit expression (*14.11*–4) for $^2\sigma(\overline{12}, 3)$:

$$g_s = \tfrac{2}{9}\,[\mu_p + 2(2\mu_n - \mu_p)] = \tfrac{2}{9}(4\mu_n - \mu_p) \quad \text{for } {}^3\text{H},$$

and

$$g_s = \tfrac{2}{9}\,(4\,\mu_p - \mu_n) \qquad \text{for } {}^3\text{He}.$$

In order to calculate the orbital gyromagnetic ratios, it is necessary to consider the angular dependence of the eigenfunctions. We choose as spatial coordinates the vectors $\vec{x}$ and $\vec{X}$, which represent, respectively, the radius vector joining the two like particles and that joining the third particle to the centre of gravity of the other two:

$$\vec{x} = \vec{x}^{(1)} - \vec{x}^{(2)}, \qquad \vec{X} = \vec{x}^{(3)} - \tfrac{1}{2}\,(\vec{x}^{(1)} + \vec{x}^{(2)}). \tag{35}$$

If $\vec{p}$, $\vec{P}$ denote the canonically conjugate momenta, it is easily verified that, in the system of reference in which the centre of gravity of the nucleus rests at the origin,

$$\hbar\,(\vec{l}^{(1)} + \vec{l}^{(2)}) = \tfrac{1}{3}\,\vec{X}\wedge\vec{P} + \vec{x}\wedge\vec{p}, \qquad \hbar\,\vec{l}^{(3)} = \tfrac{2}{3}\,\vec{X}\wedge\vec{P}. \tag{36}$$

In a state whose eigenfunction is either symmetrical or antisymmetrical with respect to $\vec{x}$ and $\vec{X}$, we have

$$\text{av}\,(\vec{X}\wedge\vec{P}) = \text{av}\,(\vec{x}\wedge\vec{p})\,;$$

combining this with the relations (36), we readily find

$$\begin{aligned} \text{av}\,(l_z^{(1)} + l_z^{(2)}) &= \tfrac{2}{3}\,\text{av}\,L_z \\ \text{av}\,l_z^{(3)} \qquad &= \tfrac{1}{3}\,\text{av}\,L_z\,. \end{aligned} \tag{37}$$

Now, the eigenfunctions of the P and D states under consideration have the symmetry property just mentioned: for (GERJUOY and SCHWINGER [42]) the angular dependence of the P wave-functions can be represented by a linear combination of the components of the only vector of even parity built up from $\vec{x}$ and $\vec{X}$, viz. $\vec{x}\wedge\vec{X}$, so that these functions are antisymmetric in $\vec{x}$ and $\vec{X}$; likewise, for the $^4D_{\frac{1}{2}}$ state, we may distinguish four distinct types of angular dependence, characterized by the tensors

$$T_{ik}^{(xX)} = \tfrac{1}{2}(x_i X_k + x_k X_i) - \tfrac{1}{3}\delta_{ik}\,|\vec{x}|\,|\vec{X}|$$

and

$$T_{ik}^{(a)} = a_i a_k - \tfrac{1}{3}\delta_{ik}\,|a|^2\,, \quad \text{with } \vec{a} = \vec{x},\ \vec{X},\ \vec{x}\wedge\vec{X},$$

from which one can form 3 symmetrical combinations $T^{(xX)}$, $T^{(\vec{x}\wedge\vec{X})}$, $T^{(x)} + T^{(X)}$ and one antisymmetrical combination $T^{(x)} - T^{(X)}$. If we accordingly introduce separately a symmetrical ^4D_s eigenfunction, comprising the three symmetrical tensors, and an antisymmetrical one, ^4D_a (so that the ^4D state

is actually described by a linear combination of ^4D_s and ^4D_a), we may thus say that the orbital Landé factors in all P and D states are $g_l = \frac{1}{3}$ for ^{3}H and $g_l = \frac{2}{3}$ for ^{3}He.

We can now write down the magnetic moments in the various component states, which enter additively into the general expression (19) for the magnetic moment, weighted by the absolute squares $|c_{PLS}|^2$ of the expansion coefficients of the ground state wave-function:

Component state	$\mu(^3\text{H})$	$\mu(^3\text{He})$
$^2\tilde{S}$	μ_p	μ_n
$^2\tilde{P}$	$-\frac{1}{3}\mu_p+\frac{8}{9}$	$-\frac{1}{3}\mu_n+\frac{4}{9}$
^2S	$\frac{1}{3}(4\mu_n-\mu_p)$	$\frac{1}{3}(4\mu_p-\mu_n)$
$^2\bar{P}$	$-\frac{1}{9}(4\mu_n-\mu_p)+\frac{2}{9}$	$-\frac{1}{9}(4\mu_p-\mu_n)+\frac{4}{9}$
^4P	$\frac{5}{9}(2\mu_n+\mu_p)-\frac{1}{9}$	$\frac{5}{9}(2\mu_p+\mu_n)-\frac{2}{9}$
$^4D_{s,a}$	$-\frac{1}{3}(2\mu_n+\mu_p)+\frac{1}{3}$	$-\frac{1}{3}(2\mu_p+\mu_n)+\frac{2}{3}$

(38)

It is essential, however, also to take account of the cross-terms which have been neglected in formula (19). These cross-terms are of two types, pertaining either to the spin or to the orbital moment, and both are represented in our case: there is clearly a non-vanishing matrix-element $(^2P|(M_z)_{\text{spin}}|\,^4P)$, and matrix-elements $(A|(M_z)_{\text{orb}}|B)$ for all pairs of states with the same spin multiplicity, but different symmetry with respect to $\vec{x}$ and $\vec{X}$: for it follows from $(A|l_z^{(1)}+l_z^{(2)}+l_z^{(3)}|B) = 0$ and from (36) that, if A and B have the same symmetry,

$$(A|l_z^{(1)}+l_z^{(2)}|B) = 2(A|l_z^{(3)}|B) = 0\,;$$

on the other hand, if A and B have different symmetry properties, one gets

$$(A|l_z^{(1)}+l_z^{(2)}|B) = -(A|l_z^{(3)}|B)\,,$$

which means that the orbital cross-terms are equal and opposite for ^{3}H and ^{3}He, so that they disappear from the sum of the magnetic moments of the two conjugate nuclei, in conformity with the general theory of *A2.221*. Likewise,

$$(^2P|\sigma_z^{(1)}+\sigma_z^{(2)}|\,^4P) = -(^2\bar{P}|\sigma_z^{(3)}|\,^4P)\,,$$

whence equal and opposite values, proportional to $\pm(\mu_p-\mu_n)$, follow for the matrix-element $(^2\bar{P}|(M_z)_{\text{spin}}|^4P)$ for ^{3}H and ^{3}He.

Using the relation $\Sigma|c_{PLS}|^2 = 1$ in order to eliminate the coefficient $c(^2\tilde{S})$, presumed to be the most important one, we may finally write the magnetic moment of ^{3}H in the form

$$\begin{aligned}\mu = \mu_p &- \tfrac{2}{3}(2\mu_p-\tfrac{1}{3})\,|c(^2\tilde{P})|^2 - \tfrac{4}{3}(\mu_p-\mu_n)\,|c(^2S)|^2 - \tfrac{4}{9}(2\mu_p+\mu_n-\tfrac{1}{2})\,|c(^2\bar{P})|^2 \\ &- \tfrac{2}{9}(2\mu_p-5\mu_n+\tfrac{1}{2})\,|c(^4P)|^2 - \tfrac{2}{3}(2\mu_p+\mu_n-\tfrac{1}{2})\,[|c(^4D_s)|^2+|c(^4D_a)|^2] \\ &+ 2\,\Re\,[c^*(^2P)\,c(^4P)\,(\mu_p-\mu_n)\,(^2\bar{P}|\sigma_z^{(3)}|\,^4P)] + 2\,\Re\,[\Sigma c^*(A)\,c(B)\,(A|l_z^{(3)}|B)]\,;\end{aligned} \qquad (39)$$

the pairs of states yielding orbital cross-terms are* $^2\widetilde{S}\,^2\overline{P}$, $^4P\,^4D_s$ and $^4D_a\,^4D_s$. A similar expression could be written down for the moment of ^{3}He; the two moments are connected by the general relation (27), which gives in this case

$$\mu(^3\mathrm{H}) + \mu(^3\mathrm{He}) = \mu_d^0 - \tfrac{2}{3}(\mu_d^0 - \tfrac{1}{2})\{3[|c(^4D_s)|^2 + |c(^4D_a)|^2] + 2[|c(^2\widetilde{P})|^2 + |c(^2\overline{P})|^2] - |c(^4P)|^2\}. \tag{40}$$

Moreover, as VILLARS [47] pointed out, it is necessary, since both moments have been measured with great precision, to take into account the small additional contributions due to the exchange term in the general expression for the magnetic moment (*A2.22*). Indeed, if the existence of such terms could be established, it would afford a very strong argument in favour of the field conception of nuclear forces. The order of magnitude of the exchange magnetic moments will depend a great deal, however, on the particular choice of the nuclear field (VILLARS [47]). Their actual calculation has been carried out by THELLUNG and VILLARS [48] on the basis of Møller and Rosenfeld's symmetrical meson theory (*16.42*), using the wave-function of the ground state of the three-nucleon system obtained by Fröhlich *et al.* (*14.22*). To this approximation, the absolute value of the effect is found to be $\mu_{\mathrm{exch}} = 0{,}023$; this quantity must be added to $\mu(^3\mathrm{H})$ and subtracted from $\mu(^3\mathrm{He})$, so that formula (40) remains valid.

Let us now proceed to the discussion of the empirical results. Formula (40) yields information of general validity about the admixtures of P and D states to the ground state. Substituting the data, one finds (ANDERSON [48])

$$3\,|c(^4D_{s,a})|^2 + 2\,|c(^2\widetilde{\overline{P}})|^2 - |c(^4P)|^2 = 0{,}13 \pm 0{,}08.$$

This relation is compatible with Gerjuoy and Schwinger's analysis of the ground state (*17.2*), according to which the admixtures of P and D_a states would be negligible, while $|c(^4D_s)|^2 \approx 0{,}04$. However, it would hardly be possible, on such assumptions, to explain the fact that the empirical value of $\mu(^3\mathrm{H})$ is *larger* than the proton moment μ_p by a factor very accurately determined as $1{,}066636 \pm 0{,}00001$. For the D_s admixture alone would not give rise to any non-diagonal contribution in the expression (39) and would consequently tend to lower the magnetic moment $\mu(^3\mathrm{H})$; to counteract this effect, the exchange contribution would have to be as large as 0,27. In view of the theoretical estimate quoted above, it does not seem very likely that a field theory accounting for the nuclear interactions would yield such a considerable exchange effect.

Whether the experimental results could be accounted for on the assumption of a more complicated mixture of orbital states, giving rise to

* The matrix-element $(^2\widetilde{S}\,|\,l_z^{(3)}\,|\,^2\widetilde{P})$ vanishes because the $^2\widetilde{S}$ eigenfunction, symmetrical in the spatial coordinates of the like particles, is an *even* function of the invariant $\vec{x}\cdot\vec{X}$. The space factor of the ^2S eigenfunction, on the other hand, is an *odd* function of this invariant.

positive cross-terms in (39), cannot be decided without an explicit determination of the eigenfunction of the ground state, much more elaborate than that attempted by Gerjuoy and Schwinger *. But it may immediately be observed that this explanation would call for the existence of spin-orbit interactions of appreciable magnitude besides the axial dipole couplings, since the latter will not to a first approximation couple a $^2\widetilde{S}$ state to any P state; this conclusion would be in agreement with the tentative interpretation of the fine structure of ^{5}He as a P doublet splitting (*17.43*).

A2.252. *The "2p shell" nuclei.* The next class of light nuclei to be considered comprises those which, on the simple picture of the quasi-atomic model, correspond to the gradual filling up of the $2p$ shell (*10.32*). The magnetic moments of the stable members of this class are all known experimentally. Since these nuclei allow of a fairly detailed treatment on the α-particle model as well as on the quasi-atomic one, they afford a very instructive comparison of these two models of nuclear structure.

The *quasi-atomic* picture offers the easier approach. In the crudest approximation, the ground state of all "2p shell" nuclei is of the ^2P type; the magnetic moment is accordingly given by (20) or (21) with (11), i.e.

$$\begin{aligned} &\text{if } J=\tfrac{3}{2}, \qquad \mu=\mu_{pn}+g_l \\ &\text{if } J=\tfrac{1}{2}. \qquad \mu=-\tfrac{1}{3}\mu_{pn}+\tfrac{2}{3}g_l\,. \end{aligned} \tag{41}$$

The calculation of the Landé factors g_l by means of formula (17) requires knowledge of the expansion coefficients b in (14). On Wigner's approximation (*9.23*), the latter quantities are independent of the particular way in which the interaction energy depends on the distances between the nucleons; they may thus be calculated for the case of the "long range fiction" (*10.31*), which reduces the work to the solution of very simple secular problems **. The results, together with the corresponding g_l values, are embodied in the following table:

A2.252 1. *Ground states and orbital Lande factors of 2p shell nuclei on Wigner's approximation*

Configuration	Ground state	Odd proton nuclei	g_l	Odd neutron nuclei	g_l
p^1 p^{11}	$(^1S\,^2P)$	^{5}Li ^{15}*N*	1	^{5}He ^{15}O	0
p^3 p^9	$\sqrt{5/9}\,(^1S\,^2P)+\sqrt{4/9}\,(^1D\,^2P)$	^{7}*Li* ^{13}N	$\frac{1}{3}$	^{7}Be ^{13}*C*	$\frac{2}{3}$
p^5 p^7	$\sqrt{5/9}\,(^1S\,^2P)-\sqrt{1/9}\,(^1D\,^2P)\pm\sqrt{1/3}\,(^1D\,^2D)$	^{9}B ^{11}*B*	$\frac{2}{3}$	^{9}*Be* ^{11}C	$\frac{1}{3}$

A symbol like $(^1S\,^2P)$ represents the eigenfunction $\Psi'_{LSL_pL_n}$ of a configuration in which the nucleons (protons or neutrons) in even number form by themselves a ^1S state, while those in odd number form a ^2P state. — The stable nuclei are in *italics*.

* See the more detailed discussion by SACHS [47*b*]. The later calculations of Mrs. GOEPPERT–MAYER and SACHS [48] ignore the possibility of spin-orbit couplings and use the inadequate form of nuclear potential mentioned in the footnote to *16.20*.

** The necessary data for setting up the secular equations for all $2p$ configurations are contained in table IV of FEENBERG and WIGNER's paper [37*a*].

In constructing this table, account has been taken (as expressed by the first column) of the symmetry between particles and "holes" resulting from the symmetry of the nuclear interaction with respect to the charge of the nucleons; only in the phases of certain component wave-functions can a change occur when passing from the one to the other of equivalent configurations: if the wave-function corresponds to an odd value of $L_p + L_n - L$, it changes sign when going over from particles to "holes". It can further be verified that the inclusion of interactions explicitly depending on the spin and isotopic coordinates gives rise only to small variations of the b's even within a wide range of variation of the interaction parameters *.

The theoretical determination of the J-values necessitates consideration of the splitting of the ground state as a result of non-central interactions. Assuming only relativistic spin-orbit coupling (ROSE and BETHE [37]), the application of Inglis' rule (*15.22*) would indicate an angular momentum $J = \frac{3}{2}$ for the nuclei with p^1, p^3 and p^5 configurations, and $J = \frac{1}{2}$ for those with p^7, p^9, p^{11} configurations: this prediction agrees with observation except in the case of ^{11}B, which very probably has an angular momentum $\frac{2}{3}$. However, one would expect the first-order non-central couplings of mixed meson theory (*16.42*) or the static axial dipole couplings given by other variants of meson theory (*16.3*, *16.41*) to be determinant in this respect, rather than the weaker relativistic couplings. In a short note, Miss PHILLIPS [40] indeed points out that the effect of the axial dipole couplings might remove the discrepancy concerning ^{11}B: the quasi-atomic model predicts above the lowest $^2P_{\frac{1}{2}}$ state two neighbouring levels with $J = \frac{3}{2}$, viz. a $^2P_{\frac{3}{2}}$ and a $^2D_{\frac{3}{2}}$ state, and the axial dipole interaction might perturb these levels to such an extent as to depress one of them below the $^2P_{\frac{1}{2}}$ state. The ground state of ^{11}B would then be a mixture of $^2P_{\frac{3}{2}}$ and $^2D_{\frac{3}{2}}$ states. On the other hand, as stated by Miss Phillips, the axial dipole coupling would in all other cases lead to the J-value expected from experiment.

If we adopt Miss Phillips' interpretation of the ground state of ^{11}B, we must get a somewhat more accurate evaluation of its magnetic moment. For the ^2D state we find the representation

$$\Psi(^2D) = \sqrt{\tfrac{7}{15}}\,(^1S\,^2D) + \sqrt{\tfrac{1}{15}}\,(^1D\,^2D) \pm \sqrt{\tfrac{7}{15}}\,(^1D\,^2P)\,, \tag{42}$$

whence, by (17),

$$g_l(^2D) = \tfrac{8}{9}\,, \tag{42a}$$

and by (19),

$$\mu(^{11}\mathrm{B}) = \mu_p + \tfrac{2}{3} - |c(^2D)|^2\,[\tfrac{8}{5}\mu_p - \tfrac{11}{15}]\,. \tag{43}$$

The experimental value,

$$\mu_{\mathrm{exp}} = \mu_p - 0{,}1\,,$$

* See ROSE and BETHE [37] and especially INGLIS [38*b*].

exhibits a radical departure from the theoretical one for a pure ^2P state; if it is used to determine the D-admixture, it yields

$$|c(^2D)|^2 \approx 0.2\,. \tag{43a}$$

On the *α-particle model*, each type of odd mass nucleus requires a special investigation in order to ascertain the quantum numbers of its ground state. The assignment of the orbital momentum Λ around the figure axis follows from symmetry considerations (*13.22*) which do not involve any detailed knowledge of the interaction energy. But the determination of J and of the possible mixture of orbital momenta $K = J \pm \frac{1}{2}$ depends essentially on the spin-orbit coupling. With the same parameter θ as in (23), the latter interaction may be written in the general form

$$\theta\alpha\vec{\Lambda}\vec{S} + (1-\theta)\beta\vec{K}\vec{S}\,, \tag{44}$$

but it is very difficult (except in trivial cases) to say anything definite about the signs and orders of magnitude of the constants α, β and θ; the only relation to be expected on general grounds between α and β is $|\beta| \ll |\alpha|$, since the former refers to the rotation of the whole nucleus, while the latter concerns the orbital motion of a single nucleon, which corresponds to larger velocities. SACHS [39], in his systematic discussion of the problem on hand, attempted to deduce the required quantities from the assumption that the spin-orbit coupling was of relativistic origin, and that the Larmor term could be neglected (*15.22*). However, INGLIS [39c] observed that in the case of ^{7}Li (*17.53*) and ^{13}C this neglect was not justified: in the first instance, on the contrary, the Larmor term is by far the larger, and in the case of ^{13}C the two terms are of the same order of magnitude and opposite sign, so that no definite prediction can be made on this basis regarding the angular momentum of this nucleus. Moreover, Inglis showed that the relativistic spin-orbit coupling in ^{11}B would be too small to bring about any appreciable admixture of $^2D_{\frac{3}{2}}$ state to the $^2P_{\frac{3}{2}}$ ground state, in contradiction (as we have just seen) to observation. It must again be pointed out that the spin-orbit couplings actually operative are undoubtedly of other origin, but nothing is known about their properties. We have therefore no other course at present but to take all J-values over from experiment, and further apply the coupling (44) with undetermined coefficients, except for whatever restrictions on their range of variation the choice of J might imply; in this sense we may still use the results of Sachs' paper.

The outcome of the discussion is summarized in table *A2.252–2* below. The simplest cases are those for which $\Lambda = 0$: since the coupling (44) becomes $\sim \vec{K}\vec{S}$, there is then no mixing of states with different K's; the expression for the magnetic moment has exactly the same form as on the quasi-atomic model, only with another numerical value of the orbital Landé factor. Analogous circumstances present themselves in the case

A2.252–2. *Ground states and magnetic moments of 2p shell nuclei according to α-particle model*

Nuclei		J (obs.)	Λ	K	μ	
^7Li	^7Be	$\frac{3}{2}$	0	1	$\mu_{pn} + g_\xi$	
^9B	^9Be	$\frac{3}{2}$	1	1	$\mu_{pn} + \frac{1}{2}(g_\xi + g_\zeta)$	if $\lvert\varepsilon\rvert \gg \lvert a\rvert$
				1,2	$\frac{3}{5}(\mu_{pn} + g_\xi + g_\zeta)$	if $\lvert\varepsilon\rvert \ll \lvert a\rvert$
^{11}B	^{11}C	$\frac{3}{2}$	1	1	$\mu_{pn} + \frac{1}{2}(g_\xi + g_\zeta)$	if $\lvert\varepsilon\rvert \gg \lvert a\rvert$
				1,2	$0{,}31\,\mu_{pn} + 0{,}76 g_\xi + 0{,}58 g_\zeta$	if $\lvert\varepsilon\rvert \ll \lvert a\rvert$
^{13}N	^{13}C	$\frac{1}{2}$	0	1	$-\frac{1}{3}\mu_{pn} + \frac{2}{3}g_\xi$	
^{15}N	^{15}O	$\frac{1}{2}$	—	1	$-\frac{1}{3}\mu_{pn} + \frac{2}{3}g_\zeta$	

$$g_\xi \approx Z/A, \qquad g_\zeta = \theta g_p + (1-\theta)\,(Z/A)$$

of ^{15}N and ^{15}O, in which, owing to the spherical symmetry, $\vec{\Lambda}$ in (44) becomes identical with $\vec{K}$; the value $\frac{1}{2}$ of J implies $a\theta + \beta(1-\theta) > 0$. In the remaining cases of mass numbers $A = 9$ and 11, the value $J = \frac{3}{2}$ gives rise to the condition $a\theta + 2\beta(1-\theta) < 0$; the mixing of states with different K-values depends on the ratio of the coefficient of spin-orbit coupling $|a|$ (the β-coupling does not contribute to the mixing) to the energy separation

$$|\varepsilon| = \left| \left\{ \frac{\hbar^2}{2I_c} K(K+1) + \tfrac{1}{2}[J(J+1) - K(K+1) - S(S+1)]\,\beta(1-\theta) \right\}_{K=J-\frac{1}{2}}^{K=J+\frac{1}{2}} \right|$$
$$= (J+\tfrac{1}{2}) \left| \frac{\hbar^2}{I_c} - \beta(1-\theta) \right| : \tag{45}$$

in the extreme case $|\varepsilon| \gg |a|$, there is no mixing; in the opposite extreme case, mixing ratios *

$$|c(^2D)|^2 : |c(^2P)|^2 = 1:3 \quad \text{for } A = 9$$
$$= 3:4 \quad \text{for } A = 11 \tag{46}$$

are found. The magnetic moment is the sum of the three contributions $\mu_s, \mu_l, \mu_\Lambda$ given by (13) with (11), (18) and (25), respectively.

We are now in a position to compare the observed magnetic moments with the predictions of both nuclear models. For this purpose, the most

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* The $c_{Km_km_s}$'s take on the following values:

	$A = 9$	$A = 11$
$c_{1,1,\frac{1}{2}} =$	$\frac{1}{2}\sqrt{3}$	$\frac{2}{\sqrt{7}}$
$c_{2,1,\frac{1}{2}} =$	$\frac{1}{2}\frac{1}{\sqrt{5}}$	$-\sqrt{\frac{3}{7}}\frac{1}{\sqrt{5}}$
$c_{2,2,-\frac{1}{2}} =$	$-\frac{1}{2}\frac{2}{\sqrt{5}}$	$+\sqrt{\frac{3}{7}}\frac{2}{\sqrt{5}}$

suitable procedure will be to analyze the measured moments into a spin and an orbital contribution in conformity with the crudest approximation of the quasi-atomic picture: this is indicated in the 3rd and 4th columns of table *A2.252–3*; the two following columns give the calculated values for the quantities in the 4th column according to both models. The case of ^{11}B has already been touched upon in the preceding discussion: we have seen how the quasi-atomic picture could be improved to explain the experimental result, and provided a sufficient margin is granted the spin-orbit couplings, the α-particle model also covers this result; whether the actual non-central forces are of the required magnitude, remains a matter for further inquiry. The cases of ^{9}Be and ^{13}C should presumably be handled in the same way as ^{11}B. The magnetic moment of ^{15}N (if the sign, as yet unknown, is taken to be negative) fits quite nicely with the unperturbed quasi-atomic picture; the α-particle model here requires the rather extreme case of purely orbital motion of the proton "hole" ($\theta = 1$) (INGLIS [39*a*]).

A2.252–3. *Observed and calculated magnetic moments of 2p shell nuclei*

Nucleus	J (obs)	Observed moment		Calculated on quasi-atomic model	Calculated on α-particte model
^{7}Li	$\frac{3}{2}$	$\mu = \mu_p +$	0,4636	0,33	0,43
^{9}Be	$\frac{3}{2}$	$\mu = \mu_n +$	0,734	0,33	0,22...1,03
^{11}B	$(\frac{3}{2})$	$\mu = \mu_p +$	−0,1036	0,67	−1,31...0,73
^{13}C	$\frac{1}{2}$	$\mu = -\frac{1}{3}\mu_n +$	0,064	0,44	0,31
^{15}N	$\frac{1}{2}$	$\mu = -\frac{1}{3}\mu_p +$	0,65*	0,67	0,31...0,67

* Assuming the minus sign

All this evidence remains vague and inconclusive. The case of ^{7}Li, on the other hand, has been studied in much greater detail. The difference between the two models is here clearly illustrated by the approximate treatment embodied in table *A2.252–3* (BETHE [38*c*]): the spin contribution is the same in both cases, but the orbital Landé factors are rather different. On the quasi-atomic model, the two p shell neutrons are paired and only the odd proton spin contributes; the orbital momentum is about equally shared between these three nucleons, the closed s shell does not participate in the orbital motion*, and hence $g_l \approx \frac{1}{3}$. On the α-particle model, the system consists of an α-cluster and a triton (^{3}H) cluster: here again, the neutrons of the triton are paired and the spin moment is that of the odd proton; but the orbital motion is now shared also by the α-cluster, so that a larger orbital gyromagnetic ratio results. Roughly, all nucleons sharing equally as in a structureless droplet, we should have $g_l = Z/A = 3/7$; somewhat more accurately, the orbital momentum would be shared between the two

* For a formal proof, taking the motion around the common centre of gravity into consideration, see ROSE and BETHE [37].

clusters in the inverse ratio of their masses, and since the specific charges of the α- and triton-clusters are $\frac{1}{2}$ and $\frac{1}{3}$, respectively, one would expect

$$g_\xi \approx \tfrac{1}{2}\cdot\tfrac{3}{7}+\tfrac{1}{3}\cdot\tfrac{4}{7}=\tfrac{17}{42}=0{,}40\,. \tag{47}$$

Although this value deviates less from the experimental one than that given by the quasi-atomic model, it is too small and there is no obvious way of improving it: INGLIS [39*a*] has verified that the polarization of the triton-cluster by the field of the α-cluster has no appreciable effect. On the other hand, various refinements of the quasi-atomic picture have not led to any substantial correction of the initial approximation. The Heisenberg and Bartlett forces would reduce the orbital Landé factor $g_l = \frac{1}{3}$ to perhaps 0,3 (ROSE and BETHE [37], INGLIS [38*b*]). The admixture to the ground state of the configuration ($^3P\,^2P$) in which the spins of the $2p$-neutrons are parallel is too small to modify the spin Landé factor significantly (INGLIS [38*b*]). The second order perturbations due to interference with excited configurations are negligible (INGLIS [38*b*]). It is not likely that the axial dipole coupling would in this case give rise to any serious perturbation of the ground state (PHILLIPS [40]). Altogether, therefore, the magnetic moment of ^{7}Li remains a bit of a puzzle.

A2.253. *The "2s shell" nuclei.* If we interpret the building up of a nucleus following the completion of the $2p$ shell as the filling of a $2s$ shell (*10.32*), we should expect the ground states of these nuclei to be of the ^2S type, with the result that the magnetic moments of the odd mass members of this class would just be μ_{pn}. The α-particle model (SACHS [39]), as shown in table *A2.253*, leads to the same prediction. The observed magnetic moment of ^{19}F reveals by its deviation from the predicted value μ_p a perturbation of the ground state. On the α-particle model, one might, following Sachs, think of an admixture of the state with $K = 1$ due to the energy terms arising from the centrifugal forces (which are usually neglected since they involve the square of the angular momentum). On the quasi-atomic model (INGLIS [39*a*]), a different interpretation suggests itself: the spin dependent interactions will bring about a small admixture of a state, belonging to a higher supermultiplet, in which the two $2s$-neutrons have parallel spins; this would give rise to a negative contribution (since $\mu_n < 0$) to the spin magnetic moment.

A2.253. *Ground states and magnetic moments of 2s shell nuclei according to α-particle model*

Nuclei	J	Λ	K	μ	μ (obs)
^{17}F ^{17}O	$\frac{1}{2}$	—	0	μ_{pn}	
^{19}F ^{19}Ne	$\frac{1}{2}$	0	0	μ_{pn}	$\mu_p - 0{,}1646$

A2.26. *Magnetic moments of stable odd nuclei.* Thanks to the rigorous expression obtainable (*A2.221*) for the magnetic moments of the stable

odd nuclei, the analysis of the empirical data in this case yields fairly precise information on the properties of the ground states of these nuclei (SACHS [46*a*]). Using the representation adapted to the quasi-atomic picture, we may say, since the angular momentum of the nuclei in question is $J = 1$, that the ground state is a mixture of ^3S, ^3D, ^1P and ^3P states. Since the P states, in the approximation of central forces, are found (*10.22*) to lie higher than the others, we may expect their proportions in the mixture to be small and even negligible for ^{2}H and ^{10}B (*10.22*). Combining formula (28) with the relation

$$|c(^3S)|^2 = 1-[|c(^3D)|^2+|c(^1P)|^2+|c(^3P)|^2] \tag{48}$$

between the expansion coefficients, one finds

$$\mu = \mu_d^0 - \tfrac{3}{2}(\mu_d^0-\tfrac{1}{2})\,[|c(^3D)|^2 + \tfrac{2}{3}\,|c(^1P)|^2 + \tfrac{1}{3}\,|c(^3P)|^2]\,, \tag{49}$$

i.e.

$$|c(^3D)|^2 + \tfrac{2}{3}\,|c(^1P)|^2 + \tfrac{1}{3}\,|c(^3P)|^2 = \beta, \qquad \beta = \frac{\mu_d^0-\mu}{\frac{3}{2}(\mu_d^0-\frac{1}{2})}. \tag{50}$$

If the P state admixture is negligible, equation (50) directly gives the ^3D-admixture. In any case, we derive from (50) upper limits for the proportions of D and P states:

$$|c(^3D)|^2 \leqslant \beta\,, \qquad |c(^1P)|^2 \leqslant \tfrac{3}{2}\,\beta\,, \qquad |c(^3P)|^2 \leqslant 3\beta\,. \tag{51}$$

From (49) and (50) we easily deduce

$$1-3\beta \leqslant |c(^3S)|^2 \leqslant 1-\beta\,. \tag{52}$$

We may repeat this argument after interchanging the roles of ^3S and ^3D, which simply means interchanging also ^1P and ^3P and replacing β by $1-\beta$; this yields the further inequalities

$$|c(^3S)|^2 \leqslant 1-\beta\,, \qquad |c(^1P)|^2 \leqslant 3\,(1-\beta)\,, \qquad |c(^3P)|^2 \leqslant \tfrac{3}{2}\,(1-\beta)\,, \tag{51a}$$

$$3\beta-2 \leqslant |c(^3D)|^2 \leqslant \beta\,, \tag{52a}$$

which are more useful when $\beta > \frac{1}{3}$. The application of these formulae to the data (table *A2.21*–1) yields the following results:

A2.26. *Ground states of stable odd nuclei*

Nucleus	$\mu_d^0-\mu$	β	$\lvert c(^3S)\rvert^2$	$\lvert c(^3D)\rvert^2$	$\lvert c(^1P)\rvert^2$	$\lvert c(^3P)\rvert^2$
^{2}D	0,023	0,04	0,96	0,04	—	—
^{6}Li	0,058	0,10	0,70...0,90	0...0,10	0...0,15	0...0,30
^{10}B	0,281	0,49	0,51	0,49	—	—
^{14}N	0,476	0,835	0...0,165	0,505...0,835	0...0,495	0...0,248

As regards the further question of the couplings responsible for admixtures of such orders of magnitude, little definite can be said. INGLIS [39*a*] has shown, in the cases of ^{6}Li and ^{14}N, that neither the Coulomb

energy, nor an amount of spin-orbit coupling sufficient to account for the separation of the ^{7}Li doublet (*17.5*) would be able to yield more than a small fraction of the observed difference $\mu_d^0 - \mu$. The axial dipole couplings are probably much more effective.

A2.261. *Unstable odd nuclei.* There is a type of nucleus not covered by the foregoing discussion, viz. the odd nuclei of mass number $4a$ and neutron excess ± 2, of which one example, the long-lived isotope ^{40}K, has been investigated experimentally as regards its magnetic moment (table *A2.21*–3). Among the "$2p$ shell" nuclei, the pairs of conjugate nuclei ^{8}Li – ^{8}B, ^{12}B – ^{12}N belong to this type. They may be treated by the same method as the odd mass nuclei on the quasi-atomic model (ROSE and BETHE [37]). The characteristics of the ^3P ground states are as follows:

Configuration		Ground state			g_l			g_l
p^4	p^8	$\pm\sqrt{\frac{5}{8}}\,(^2P^2P) + \sqrt{\frac{3}{8}}\,(^2D^2P)$	^{8}Li	^{12}N	$\frac{1}{8}$	^{8}B	^{12}B	$\frac{7}{8}$

According to (19) and (12), the magnetic moments are given by

$$\mu = \tfrac{1}{2} J (\mu_d^0 + g_l)\,. \tag{53}$$

Relativistic spin-orbit coupling (limited to the Thomas effect) indicates $J = 2$ for ^{8}Li – ^{8}B and $J = 0$ for ^{12}N – ^{12}B; but these J-values are, of course, very uncertain.

The case of ^{40}K is rather strange (INGLIS [41]). From the possible components of the ground state:

$$^1G_4,\quad {}^3FGH_4,\quad {}^5DFGHI_4,\quad \text{etc.},$$

the singlet and triplet ones are both excluded (as predominant components), because they would most probably imply a positive magnetic moment: this is obvious for the $^1G_4$ state, since g_l will certainly be positive; and if we assume that in the triplet states g_s is $\approx \mu_d^0$ as for the light odd nuclei, we shall have in the most favourable case, viz. that of a $^3H_4$ state, $\mu = \frac{4}{5}(6g_l - g_s)$, which is positive for any plausible value of g_l. For quintet states, we expect the gyromagnetic ratio g_s to be made up of one proton and 3 neutron contributions, i.e.

$$g_s = \tfrac{1}{2}(\mu_p + 3\mu_n) = -1{,}47\,, \tag{54}$$

whereas g_l might be $\approx \frac{1}{4}$, corresponding to equal sharing of the orbital momentum between those four nucleons. Then, two of the possible quintet states, viz. $^5D_4$ and $^5F_4$, are found, by (19), to give negative values of acceptable order of magnitude for the magnetic moment:

$$\begin{aligned} \mu(^5D_4) &= 2\,(g_s + g_l) = -2{,}44 \\ \mu(^5F_4) &= \tfrac{7}{5} g_s + \tfrac{13}{5} g_l = -1{,}41\,. \end{aligned} \tag{55}$$

We are thus driven to the conclusion that the ground state of ^{40}K would probably be a mixture of the quintet states $^5D_4 + {}^5F_4$, although there is no obvious reason for the lack of pairing of two neutrons in this case. This difficulty does not seem to arise in the case of the heavier odd nucleus ^{176}Lu, whose ground state could well be a triplet one (INGLIS [41]).

A2.27. *Relativistic correction to the magnetic moment.* The preceding considerations have been based on the non-relativistic expression for the magnetic moment and although there is little doubt that this approximation is quite sufficient, it may be of some interest to estimate the order of magnitude of the relativistic correction. About the exact form of the magnetic moment operator there is some uncertainty owing to the occurrence of the anomalous moments of proton and neutron. However, if we accept the principle of the explanation of these moments on field theory, we must regard them as originating from a definite current distribution (cf. *6.13*) and we then simply get, for the magnetic moment of a nuclear system *,

$$\vec{M} = \sum_i \varrho_3^{(i)} \left(\vec{\mu}^{(i)} + \tau_-^{(i)} \vec{l}^{(i)} \mu_0\right); \tag{56}$$

this formula differs from the non-relativistic expression (1), (2), (3) by the occurrence of the operators $\varrho_3^{(i)}$, implying corrections of the second order in the nucleon velocities. We shall carry out the computation of this correction in two cases:

(1) one nucleon moving in a central field: this may be taken as a crude picture of the situation obtaining in odd mass nuclei (*A2.23*);

(2) the ground state of the deuteron.

Nucleon in central field. The well-known solution of Dirac's equation in this case ** may be written in our notation (*4.21*):

$$\psi = w_+(\varrho_3') \psi_+ + w_-(\varrho_3') \psi_-;$$

$l = j - \frac{1}{2}$:

$$\psi_+ = \frac{1}{\sqrt{2l+1}} \left[\sqrt{l+m+\tfrac{1}{2}}\, Y_l^{m-\frac{1}{2}} v_+(\sigma_z') - \sqrt{l-m+\tfrac{1}{2}}\, Y_l^{m+\frac{1}{2}} v_-(\sigma_z')\right] F_+(r)$$

$$\psi_- = \frac{i}{\sqrt{2l+3}} \left[\sqrt{l-m+\tfrac{3}{2}}\, Y_{l+1}^{m-\frac{1}{2}} v_+(\sigma_z') + \sqrt{l+m+\tfrac{3}{2}}\, Y_{l+1}^{m+\frac{1}{2}} v_-(\sigma_z')\right] F_-(r); \tag{57}$$

$l = j + \frac{1}{2}$:

$$\psi_+ = \frac{1}{\sqrt{2l+1}} \left[\sqrt{l-m+\tfrac{1}{2}}\, Y_l^{m-\frac{1}{2}} v_+(\sigma_z') + \sqrt{l+m+\tfrac{1}{2}}\, Y_l^{m+\frac{1}{2}} v_-(\sigma_z')\right] F_+(r)$$

$$\psi_- = \frac{i}{\sqrt{2l-1}} \left[\sqrt{l+m-\tfrac{1}{2}}\, Y_{l-1}^{m-\frac{1}{2}} v_+(\sigma_z') - \sqrt{l-m-\tfrac{1}{2}}\, Y_{l-1}^{m+\frac{1}{2}} v_-(\sigma_z')\right] F_-(r).$$

From the wave-equation, involving the potential energy $\mathcal{V}(r)$, one readily derives the relation

$$\int (E - \mathcal{V} + M) |\psi_-|^2 \, dv = \int (E - \mathcal{V} - M) |\psi_+|^2 \, dv \tag{58}$$

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* See, e.g., MOLLER and ROSENFELD [43], p. 41 sq. In formula (37) of this paper, however, the factors $\varrho_3^{(i)}$ are wrongly omitted from the terms containing the anomalous moments.

** See, e.g., BETHE [33], p. 311 sq.

between large and small components ψ_+, ψ_-; for not too high velocities, the left-hand side can be replaced by $2M \int |\psi_-|^2 dv$, so that

$$\int |\psi_-|^2 dv \approx \frac{1}{2M} \int (E - V - M)(|\psi_+|^2 + |\psi_-|^2)\, dv\,;$$

the integral on the right represents the average kinetic energy of the nucleon in the stationary state ψ. Denoting by β^2 the ratio of this quantity to the rest energy M, we thus have

$$\int |\psi_-|^2 dv \approx \tfrac{1}{2}\beta^2\,, \tag{59}$$

to which we may add the normalization condition in the form

$$\int |\psi_+|^2 dv = 1 - \tfrac{1}{2}\beta^2\,. \tag{60}$$

With the help of these formulae, it is an easy matter to calculate the magnetic moment. One finds

$l = j - \frac{1}{2}$:

$$\text{av}\, \varrho_3\, \sigma_z = \frac{m}{j}\left[1 - \frac{1}{2l+3}\beta^2\right]$$

$$\text{av}\, \varrho_3\, l_z = \frac{m}{j}\left[l - \frac{2l^2+4l+1}{2l+3}\beta^2\right];$$

$l = j + \frac{1}{2}$: (61)

$$\text{av}\, \varrho_3\, \sigma_z = -\frac{m}{j+1}\left[1 + \frac{1}{2l-1}\beta^2\right]$$

$$\text{av}\, \varrho_3\, l_z = \frac{m}{j+1}\left[l + 1 - \frac{2l^2-1}{2l-1}\beta^2\right].$$

Hence, the correction $\Delta\mu$ to be added to the non-relativistic expression for the magnetic moment μ is, for both values of $l = j \pm \frac{1}{2}$,

$$\Delta\mu = -\frac{1}{j+1}\beta^2\left[\tfrac{1}{2}\mu_{pn} + (j^2 + j - \tfrac{1}{4})\, g_p\right]. \tag{62}$$

For odd mass nuclei, this would mean a correction of the order of 1 %.

In treating this problem, MARGENAU [40*a*] and CALDIROLA [46] prefer to compute directly the average value, in the state (57), of the interaction energy of the nucleon with a constant external magnetic field. Margenau's statement that the calculation of the average value of the operator (1), (2), (3) gives a different answer is, of course, formally correct, but he seems to have overlooked the fact that this operator is not the true expression for the magnetic moment. If the factors $\varrho_3^{(i)}$ are introduced, as in (56), no discrepancy arises. Margenau's treatment of the anomalous moments is thus necessarily inadequate; in this respect, Caldirola indicates the right procedure, which consists in starting from the Hamiltonian (*4.221*–9), in which, however, a factor ϱ_3 must be inserted in the last term.

Ground state of the deuteron *. Using the representation (*4.33*–18)

* The following calculation is inaccurate inasmuch as it neglects the second order correction to the Hamiltonian of the deuteron problem (*15.22*), and it accordingly does not

of the eigenfunction of any stationary state of the deuteron, we must in the first place fix its normalization. Multiplying it by some suitable constant C, we may write down the normalization condition, exact to the second order in the nucleon velocities,

$$|C|^2 [\int |\overset{(0)}{\psi}|^2 + \int |\overset{(1)}{\psi}|^2] = 1\,. \tag{63}$$

We may now take for $\overset{(0)}{\psi}$ and $\overset{(1)}{\psi}$ the expressions (*4.34–29b*), in which the radial functions are normalized to unity. Thus, $\int |\overset{(0)}{\psi}|^2 = 1$, and for $\int |\overset{(1)}{\psi}|^2$ we get for instance in the case $l = j - 1$:

$$\begin{aligned}
\int |\overset{(1)}{\psi}|^2 &= \frac{\hbar^2}{2M^2}\int_0^\infty \left[\frac{d}{dr}\,{}^3R^{(l)} - \frac{j}{r}\,{}^3R^{(l)}\right]^2 dr \\
&= -\frac{\hbar^2}{2M^2}\int_0^\infty {}^3R^{(l)}\left[\frac{d^2}{dr^2}\,{}^3R^{(l)} - \frac{j^2}{r^2}\,{}^3R^{(l)} + \frac{2j}{r}\frac{d}{dr}\,{}^3R^{(l)}\right] dr \\
&= -\frac{\hbar^2}{2M^2}\int_0^\infty {}^3R^{(l)}\left[\frac{d^2}{dr^2}\,{}^3R^{(l)} - \frac{l(l+1)}{r^2}\,{}^3R^{(l)}\right] dr \\
&= \frac{1}{2M}\int_0^\infty {}^3R^{(l)}\,({}^3\varepsilon^{(l)} - {}^3\mathcal{V}^{(l)})\,{}^3R^{(l)}\,dr = \beta^2\,,
\end{aligned} \tag{64}$$

in virtue of the wave-equation (*4.331–27*). The quantity β^2 represents the ratio of the mean kinetic energy *per nucleon* to the rest energy of a nucleon. The other cases yield the same result, so that we get from (63)

$$|C|^2 \approx 1 - \beta^2\,. \tag{65}$$

Now, for the expectation value of any operator of the form

$$A = \varrho_3^{(1)} A^{(1)} + \varrho_3^{(2)} A^{(2)},$$

in which the A's do not contain the ϱ's we get, again neglecting a term of higher order,

$$\begin{aligned}
\text{av}\, A = |C|^2 [\int {}^3f_1^* \,(A^{(1)} + A^{(2)})\,{}^3f_1 + \int {}^3f_0^* \,(A^{(1)} - A^{(2)})\,{}^1f_0 \\
+ \int {}^1f_0^* \,(A^{(1)} - A^{(2)})\,{}^3f_0]\,;
\end{aligned}$$

calling $\overset{(0)}{A}$ the non-relativistic approximation, given by the first integral

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claim to be more than an estimate of order of magnitude. Calculations taking account of this correction have been carried out, for the ^3S state, by BREIT and BLOCH [47c]. Adopting the Hamiltonian (*15.22–16*), they find an additional term $+\frac{1}{6}\beta^2$ to the first formula (70); with the Hamiltonian (*15.22–17*), the additional term is $-\varepsilon_0/M + \frac{1}{2}\beta^2$. Similar calculations by SACHS [*47a*] are incorrect in so far as they do not use the complete second order Hamiltonians.

SACHS [*47a*] and PRIMAKOFF [47] discuss the influence of axial dipole forces on the relativistic correction to the magnetic moment.

between the brackets, we may write the relativistic correction, up to the second order in the velocities, in the form

$$\Delta A = -\beta^2 \overset{(0)}{A} + 2\Re \int {}^3f_0^* \, (A^{(1)} - A^{(2)}) \, {}^1f_0 \,. \tag{66}$$

In all states with $l = j$, the last term of this expression vanishes. In the states with $l = j \pm 1$, this term involves a singlet-triplet transition and therefore vanishes, in particular, for all operators independent of the spin coordinates. For the magnetic moment of the deuteron, we have

$$\Delta \mu = -\beta^2 \overset{(0)}{\mu} + \Delta\mu' \,, \tag{67}$$

the term $\Delta\mu'$ arising from the spin term only. Since

$$\mathrm{av}\; \mathsf{T}_+^{(i)} = \mathrm{av}\; \mathsf{T}_-^{(i)} = \tfrac{1}{2} \,, \tag{68}$$

we get

$$\Delta\mu' = \mu_d^0 \, \Re \, [\int {}^3f_0^* \, (\sigma_z^{(1)} - \sigma_z^{(2)}) \, {}^1f_0]_{m=j} \,,$$

for which a straightforward calculation, using (64), yields

$$\Delta\mu' = \pm \, \mu_d^0 \, \beta^2 \, \frac{2j}{2j+1} \,; \tag{69}$$

the $\pm$ signs correspond to $l = j \mp 1$, respectively. In particular, the corrections for the components of the ground state are

$$\begin{aligned} \Delta\mu({}^3S_1) &= -\tfrac{1}{3}\, \mu_d^0 \, \beta^2 \\ \Delta\mu({}^3D_1) &= -\tfrac{3}{4}\, \beta^2 - \tfrac{1}{6}\, \mu_d^0 \, \beta^2 \,. \end{aligned} \tag{70}$$

The order of magnitude of β^2 is 1 %.

A2.3. Quadrupole moments of nuclei

A2.31. *The empirical data.* Electric quadrupole moments have been determined for a large number of odd mass nuclei; in addition, the electric quadrupole moments of two odd nuclei, the stable ^{2}D and the long-lived ^{176}Lu are known. Recently, the analysis of the hyperfine structure of microwave molecular spectra has revealed the existence of quadrupole moments of ^{14}N (SIMMONS and GORDY [48] *, GORDY *et al.* [48*a*], TOWNES *et al.* [47*a*]) and of the ^{35,37}Cl and ^{79,81}Br isotopes (TOWNES *et al.* [47*a*], GORDY *et al.* [47*b*, *c*]). By this method, the absolute values of the quadrupole moments cannot be ascertained without further study, since the quantity measured is the product (*A2.11*–6) of the quadrupole moment by the gradient of the electric field; only the signs and the ratios of the moments of isotopes can be given directly. However, for heavier nuclei, an estimate of the field gradient, and consequently of the quadrupole moment itself, can be inferred from data of atomic spectroscopy

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* This paper contains references to previous work, experimental and theoretical. For the theory of the effect, including second order corrections, see further BARDEEN and TOWNES [48]. The authors do not all use the same convention as regards the definition of the quadrupole coupling; see a clarifying note by FELD [47].

(TOWNES [47*b*]); this method involves the neglect of the distortion of the electronic clouds due to molecular binding. Its application to the case

A2.31. *Electric quadrupole moments of nuclei*

Nucleus	J	μ	Q 10^{-24} cm²	Nucleus	J	μ	Q 10^{-24} cm²
			Odd mass nuclei				
	Odd proton nuclei				*Odd neutron nuclei*		
$^{35}_{17}Cl$	$\frac{5}{2}$ *	1,368	$-0{,}0795$				
$^{37}_{17}Cl$	$\frac{5}{2}$ *	1,136	$-0{,}062$				
$^{63}_{29}Cu$	$\frac{3}{2}$	2,227	$-0{,}1$				
$^{65}_{29}Cu$	$\frac{3}{2}$	2,385	$-0{,}1$				
$^{69}_{31}Ga$	$\frac{3}{2}$	1,994	0,186				
$^{71}_{31}Ga$	$\frac{3}{2}$	2,540	0,117				
$^{75}_{33}As$	$\frac{3}{2}$	1,6	0,3				
$^{79}_{35}Br$	$\frac{3}{2}$	2,107	$Q \approx 0{,}2$				
$^{81}_{35}Br$	$\frac{3}{2}$	2,271	$0{,}77 \cdot Q$				
				$^{83}_{36}Kr$	$\frac{9}{2}$	$-1{,}0$	0,15
$^{115}_{49}In$	$\frac{9}{2}$	5,49	0,84				
$^{127}_{53}I$	$\frac{5}{2}$	2,8	0,8?				
				$^{131}_{54}Xe$	$\frac{3}{2}$	0,7	$0 \pm 0{,}1$
$^{151}_{63}Eu$	$\frac{5}{2}$	3,4	1,2				
$^{153}_{63}Eu$	$\frac{5}{2}$	1,5	2,5				
				$^{173}_{70}Yb$	$\frac{5}{2}$	$-0{,}65$	3,9
$^{175}_{71}Lu$	$\frac{7}{2}$	2,6	5,9				
$^{185}_{75}Re$	$\frac{5}{2}$	3,3	2,8				
$^{187}_{75}Re$	$\frac{5}{2}$	3,3	2,6				
				$^{201}_{80}Hg$	$\frac{3}{2}$	$-0{,}6$	0,5
$^{209}_{83}Bi$	$\frac{9}{2}$	3,6	$-0{,}4$				
			Odd nuclei				
	Stable odd nuclei				*Unstable odd nuclei*		
$^{2}_{1}D$	1	0,857	$2{,}73 \cdot 10^{-3}$				
$^{14}_{7}N$	1	0,403	0,01...0,1	$^{176}_{71}Lu$	$\gtrsim 7$	3,8	7

* From molecular data, however, TOWNES *et al.* [47*a*], derive a J-value $\frac{3}{2}$

of NH_3, from which the quadrupole moment of ^{14}N should be derived, is therefore rather uncertain, and only approximate limiting values can be indicated *. On the other hand, the quadrupole moments of the $^{35,\,37}Cl$

* For the deuteron case, see footnote to (*6.12–7*).

isotopes have lately been measured by an atomic beam method (DAVIS *et al.* [48]); this would enable us in this case to derive from the molecular data some direct information about the field gradients in the various molecular structures studied. All available results are collected in table *A2.31*; references to the sources will be found in INGLIS' oft-quoted paper [41] *. It is a most striking feature of this material that *almost all quadrupole moments are positive*, indicating an electricity distribution elongated along the axis of angular momentum of the nucleus. Moreover, *with increasing charge number, the values of the quadrupole moments pass through a maximum at about* $Z = 71$.

In addition, it has been verified (DAILEY *et al.* [46], TOWNES *et al.* [47c]) that, in accordance with the theory (*A2.13*), the nuclei ^{13}C, ^{15}N, ^{33}S, for which $J = \frac{1}{2}$, and ^{34}S, with $J = 0$, have no quadrupole moment.

A2.32. *Quadrupole moments of odd mass nuclei and nuclear models.* It is obvious that the elongated form of many nuclei revealed by the positive quadrupole moments is incompatible with the *droplet model*, according to which the only shape stable against arbitrary distortions is that of an oblate ellipsoid of revolution around the axis of angular momentum. Moreover, the quantitative discussion carried out by Miss WAY [39a] shows that the flattening due to rotation would be much too small to account for the absolute value of the negative quadrupole moment of ^{209}Bi. Just as the magnetic moment, the electric quadrupole is an essentially structural property of the nucleus. And just as in the case of the magnetic moment, it is, as emphasized by INGLIS [41], the *α-particle model* that affords the best possibilities of interpretation.

In the first place, the investigation of the stablest packings of hard spheres (WEFELMEIER [37a]) suggests that the most elongated framework of α-particles is in the neighbourhood of $Z = 71$, just where the largest quadrupole moments are observed. However, it is essential for this interpretation that the total angular momentum should orient itself along the longitudinal axis of the framework. Now, the evidence from the magnetic moments suggests, as we have seen, that the ground state of an odd mass nucleus is a doublet state, in which the direction of the total angular momentum coincides with the average direction of the orbital momentum; further, the latter seems to be largely ascribable to the motion of the odd nucleon. As pointed out by FANO [37], the exclusion principle entails a repulsion of a nucleon by an α-cluster, and consequently a concentration of the wave-function of the odd nucleon as far away as possible from the concentrations of α-clusters: for an elongated structure of the α-clusters, this means an orientation of the orbital momentum of the odd nucleon along the longitudinal axis, as required for the explanation of the positive

* All the quadrupole moments given in the table, with the exception of those of $^{79,81}Br$, ^{2}H and ^{14}N, have been determined directly by methods involving only the atoms of the elements concerned. For the $^{69,71}Ga$ isotopes, the recent results of BECKER and KUSCH [48] are given. Concerning ^{127}I, see also GORDY *et al.* [48b].

quadrupole moments. The same argument applied to a flattened structure would lead to an orientation of the orbital momentum perpendicular to the figure axis; taking the zero point rotation into consideration, this situation would correspond to spherical symmetry of the average charge distribution and a vanishing quadrupole moment.

The fact that negative quadrupole moments occur only among odd proton nuclei is also in harmony with the picture derived from the interpretation of the magnetic moments. For the motion of single proton with a non-vanishing orbital momentum L_p in a central field gives rise to a negative quadrupole moment (BETHE [33], p. 557)

$$Q_{L_p} = (Q_{L_p m_p})_{m_p = L_p} = -\frac{2L_p}{2L_p+3} \operatorname{av} r^2, \tag{1}$$

where

$$Q_{L_p m_p} = -\frac{2[3m_p^2 - L_p(L_p+1)]}{(2L_p-1)(2L_p+3)} \operatorname{av} r^2; \tag{2}$$

and this remains true for heavier nuclei, as we shall see presently (*A2.321*), when the interaction with the other constituent nucleons is taken into account. If the framework of α-particles is nearly spherically symmetrical, the negative quadrupole moment of the odd proton may determine the sign of the whole effect. This simple interpretation, however, does not seem immediately applicable to the five known cases of nuclei with negative quadrupole moments, because it is not clear that their α-framework should have nearly spherical symmetry. (For the detailed discussion, see INGLIS [41].)

A2.321. *Quadrupole moment due to a single proton.* The preceding discussion must be completed by the proof of the property just enunciated: *in heavier nuclei, the motion of a single proton gives rise to a negative quadrupole moment.* This proof has been given by WELLES [42]. Assuming the ground state to belong to the lowest supermultiplet, we have $S = \frac{1}{2}$ and $L = J \pm \frac{1}{2}$. The substate with $m = J$ corresponds, for $L = J - \frac{1}{2}$, to definite values $m_l = L$ and $m_s = \frac{1}{2}$ of the z-components L_z and S_z; but for $L = J + \frac{1}{2}$, we have a mixture * of two states with $m_l = L$, $m_s = -\frac{1}{2}$ and $m_l = L - 1$, $m_s = +\frac{1}{2}$:

$$\Psi_{L=J+\frac{1}{2};S=\frac{1}{2}} = \sqrt{\frac{2J+1}{2J+2}}\, \Psi_{L=J+\frac{1}{2};m_l=J+\frac{1}{2};m_s=-\frac{1}{2}} - \frac{1}{\sqrt{2J+2}}\, \Psi_{L=J+\frac{1}{2};m_l=J-\frac{1}{2};m_s=\frac{1}{2}}\,. \tag{3}$$

If we therefore write the eigenfunction of the ground state (with $m = J$) in the form

$$\Psi = c_{J-\frac{1}{2}}\, \Psi_{L=J-\frac{1}{2};S=\frac{1}{2}} + c_{J+\frac{1}{2}}\, \Psi_{L=J+\frac{1}{2};S=\frac{1}{2}}\,. \tag{4}$$

* See E. CONDON and G. SHORTLEY, *The theory of atomic spectra* (1935), Chapt. V, 4, formula (8*b*), p. 123.

we have for the quadrupole moment

$$Q = |c_{J-\frac{1}{2}}|^2 \, Q_{J-\frac{1}{2}} + |c_{J+\frac{1}{2}}|^2 \left[\frac{2J+1}{2J+2} \, Q^+_{J+\frac{1}{2}} + \frac{1}{2J+2} \, Q^-_{J+\frac{1}{2}}\right] - 2\,\Re\, c^*_{J-\frac{1}{2}} \, c_{J+\frac{1}{2}} \frac{1}{\sqrt{2J+2}} \, q_J \,. \tag{5}$$

In this formula, the average values $Q_{J-\frac{1}{2}}$, $Q^+_{J+\frac{1}{2}}$, $Q^-_{J+\frac{1}{2}}$ are the diagonal elements of the operator $3z^2 - r^2$ of the odd proton with respect to the wave-functions $\Psi_{Lm_l m_s}$ with $L = m_l = J - \frac{1}{2}$, $L = m_l = J + \frac{1}{2}$ and $L = J + \frac{1}{2}$, $m_l = J - \frac{1}{2}$; the non-diagonal element

$$q_J = \int \Psi^*_{L=J-\frac{1}{2}; m_l=J-\frac{1}{2}; m_s=\frac{1}{2}} \, (3z^2 - r^2) \, \Psi_{L=J+\frac{1}{2}; m_l=J-\frac{1}{2}; m_s=\frac{1}{2}} \tag{6}$$

is generally negligible.

The computation of the Q's necessitates a further decomposition of the eigenfunctions. In the first place, we write

$$\Psi_{Lm_l m_s} = \Sigma \, b_{Lm_l m_s; L_p L_n} \, \Psi_{Lm_l; L_p L_n} \,, \tag{7}$$

L_p being the orbital quantum number of the odd proton, L_n that of the "core" formed by the neutrons and the other protons (assumed to give rise to a spherically symmetrical charge distribution). Further,

$$\Psi_{Lm_l; L_p L_n} = \sum_{m_p + m_n = m_l} (Lm_l \,|\, m_p m_n)_{L_p L_n} \, \varphi_{L_p m_p} \, \chi_{L_n m_n} \,, \tag{8}$$

where φ is a function of the coordinates of the odd proton alone, and χ a function of the coordinates of all the other particles. The expansion coefficients $(Lm_l \,|\, m_p m_n)_{L_p L_n}$ can be derived by arguments from group theory *. With the expansion (7), the Q's take the form

$$Q_{Lm_l} = \Sigma \, |b_{Lm_l m_s; L_p L_n}|^2 \, Q_{Lm_l; L_p L_n} + 2\,\Re\, \Sigma \, b^*_{Lm_l m_s; L_p \pm 2, L_n} \, b_{Lm_l m_s; L_p L_n} \, q^{\pm}_{L_p} \,, \tag{9}$$

with diagonal elements $Q_{Lm_l; L_p L_n}$ with respect to the $\Psi_{Lm_l; L_p L_n}$ and non-diagonal elements $q^{\pm}_{L_p}$ corresponding to a change from $L_p \pm 2$ to L_p. Again, the contributions from the latter elements can be shown to be small owing to compensation of terms of both signs. The diagonal elements, according to (8), are given by

$$Q_{Lm_l; L_p L_n} = \sum_{m_p + m_n = m_l} |(Lm_l \,|\, m_p m_n)_{L_p L_n}|^2 \, Q_{L_p m_p} \,, \tag{10}$$

with $Q_{L_p m_p}$ defined by (2). The average $\mathrm{av}\, r^2$ occurring in this last quantity can be estimated, e.g. by assuming the odd proton to move in some fictitious central field, as usual in the quasi-atomic model; the order of magnitude of $\mathrm{av}\, r^2$ is, of course, that of the square of the nuclear radius

$$R^2 = r_0^2 \, A^{2/3} \approx 2A^{2/3} \cdot 10^{-26} \, \mathrm{cm}^2.$$

* They are given, e.g., by E. CONDON and G. SHORTLEY, *op. cit.* Chapt. III, *14*.

It will be observed that the $Q_{L_p m_p}$ occurring in the average quadrupole moment (10) of a proton of given orbital momentum in the nucleus may in general be of both signs, so that, in contrast to the case of an isolated proton, the average $Q_{Lm_l;L_pL_n}$ can be positive as well as negative. However, if we make the assumption, consonant with the droplet model (*A2.22–22*), that all $|b_{Lm_lm_s;L_pL_n}|^2$ are approximately the same, the overall averages $Q_{L;m_l=L}$ turn out, on actual calculation, to be always negative, though smaller in absolute value than the single proton moment (1). As regards the average $Q^-_{J+\frac{1}{2}}$, calculation shows that it is related to $Q^+_{J+\frac{1}{2}}$ by the simple equation

$$Q^-_{J+\frac{1}{2}} = \frac{2J-5}{2J+1}\, Q^+_{J+\frac{1}{2}}, \tag{11}$$

so that the total contribution corresponding to $L = J + \frac{1}{2}$ in (5) is

$$\frac{(2J-1)(J+2)}{(2J+1)(J+1)}\, Q^+_{J+\frac{1}{2}}; \tag{12}$$

the coefficient of $Q^+_{J+\frac{1}{2}}$ is always positive, except for $J = \frac{1}{2}$, when it vanishes, as it should (*A2.13*). Welles' final results are as follows:

J	$Q/\text{av}\, r^2$
$\frac{3}{2}$	$-0{,}057\,(1+0{,}72\,\lvert c_{J+\frac{1}{2}}\rvert^2)$
$\frac{5}{2}$	$-0{,}14\ \ (1+0{,}14\,\lvert c_{J+\frac{1}{2}}\rvert^2)$
$\frac{7}{2}$	$-0{,}19\ \ (1+0{,}31\,\lvert c_{J+\frac{1}{2}}\rvert^2)$
$\frac{9}{2}$	$-0{,}27\ \ (1+0{,}2\ \lvert c_{J+\frac{1}{2}}\rvert^2)$

(13)

In particular, one gets

$$\begin{aligned} &\text{for Cu:} \quad Q \approx -0{,}05\,(1+0{,}72\,|c_{J+\frac{1}{2}}|^2)\,10^{-24}\,\text{cm}^2 \\ &\text{for Bi:} \quad Q \approx -0{,}33\,(1+0{,}2\,|c_{J+\frac{1}{2}}|^2)\,10^{-24}\,\text{cm}^2; \end{aligned} \tag{14}$$

according to these figures, the observed negative quadrupole moments might well be due to the motion of a single proton around a spherically symmetrical core.

A2.322. *Quadrupole moments of light nuclei.* If we adopt quasi-atomic model representations of the ground states of light nuclei, such as those given in table *A2.252–1* for the odd mass $2p$ shell nuclei and in *A2.261* for the unstable odd nuclei of this class, we can compute accurately the quadrupole moments of such nuclei. This has been done by WELLES [42]. Far from being negligible, the cross-terms analogous to the $\dot{q}^\pm_{L_p}$ of formula (9) above are often predominant; but this is a feature of the quasi-atomic model, connected with the small number of component wave-functions entering into the representation of the ground state. Configurations symmetrical with respect to particles and "holes", but corresponding to the same value of the angular momentum yield, of course, quadrupole moments of opposite signs and nearly equal absolute values (the difference arising only from the slightly unequal av r^2): these circum-

stances occur for the pairs $^9B - ^{11}B$, $^9Be - ^{11}C$. Table *A2.322* summarizes the results obtained by Welles:

A2.322. *Calculated quadrupole moments of light nuclei*

Nucleus	J	Q 10^{-26} cm^2	Nucleus	J	Q 10^{-26} cm^2
^7Li	$\frac{3}{2}$	$-2{,}7$	^7Be	$\frac{3}{2}$	$-5{,}7$
^9B	$\frac{3}{2}$	$-1{,}2$	^9Be	$\frac{3}{2}$	$3{,}4$
^{11}B	$\frac{3}{2}$	$1{,}3$	^{11}C	$\frac{3}{2}$	$-3{,}9$
^{13}N	$\frac{1}{2}$	0	^{13}C	$\frac{1}{2}$	0
^8Li	2?	$1{,}4$	^8B	2?	$7{,}6$
^{12}N	0?	0	^{12}B	0?	0

A2.33. *Quadrupole moments of stable odd nuclei.* Like the deuteron, the other stable odd nuclei should exhibit quadrupole moments, increasing in magnitude as the mass number increases, owing to the increasing amount of D (and perhaps P) state entering into the representation of the ground state (*A2.26*). The expected order of magnitude would be roughly $10^{-27} \ldots 10^{-26}$ cm^2 (as for other light nuclei; cf. table *A2.322*). Hitherto, we have only some indication about the quadrupole moment of ^{14}N (*A2.31*).

APPENDIX III

STRONG COUPLING THEORY OF NUCLEAR FORCES

A3.0. For a first introduction to the subject of strong coupling theory, PAULI's booklet *Meson theories of nuclear forces* [46] can be recommended; further, section 4 of WENTZEL's article *Recent research in meson theory* [47] gives an authoritative survey of the question, with full references to the original papers. The following notes have no other purpose but to mention the main points of interest.

A3.1. Excited states of nucleons

A3.11. *Theory.* The Hamiltonian of a single nucleon at rest consists of a term describing the pure meson field created by it, and another expressing the interaction between this field and the nucleon. The latter contains a source density involving a spatial distribution function $D(P)$ (*1.35*) and some operator depending on the spin and isotopic variables $\vec{\sigma}, \boldsymbol{\tau}$ of the nucleon. Likewise, the operator defining the angular momentum of the nucleon is of the form

$$\vec{s} = \tfrac{1}{2}\,\vec{\sigma} + \vec{s}_{\text{field}}, \tag{1}$$

the term $\vec{s}_{\text{field}}$, which represents the angular momentum of the meson field, being a function of the field variables alone; a similar operator in isotopic space may be introduced:

$$\mathbf{t} = \tfrac{1}{2}\,\boldsymbol{\tau} + \mathbf{t}_{\text{field}}, \tag{2}$$

by means of which the charge of the nucleon (in units e) is expressed as $\frac{1}{2} - t_3$.

The fundamental problem consists in finding a canonical transformation which reduces the Hamiltonian to the diagonal form. More generally, we may assume the field originally to include also free mesons: the transformation in question then effects a separation between the proper meson field of the nucleon and that representing the free mesons. The eigenvalues of the total Hamiltonian are obtained in the form of an expansion with respect to decreasing powers of the coupling constant g (*1.35*); a constant term in g^2 is interpreted as the self-energy of the nucleon; terms independent of g pertain to the "free" meson field, distorted by the presence of the nucleon, and thus include a description of the scattering of mesons by the nucleon; finally, a term in g^{-2} occurs, representing the excitation energy of the nucleon due to the inertia of its spin and charge. This last term has the simple form of the proper energy of a symmetrical top, such that the components of $\vec{s}$ correspond to the projections of the angular momentum of the top on axes fixed in space, while those of $\mathbf{t}$ play the

ne part as the projections of the angular momentum on axes rigidly ed in the top. Denoting by j the quantum number of the angular ımentum ($j = \frac{1}{2}, \frac{3}{2}, \ldots$) and by n an eigenvalue of its projection t_3 on the ɔdy-fixed" axis 3 ($|n| \leq j$), we obtain the energy values

$$E_{j,n} = \text{const} + \tfrac{1}{2}\,\varepsilon\,[2j(j+1) - n^2] \tag{3}$$

the case of a charged meson theory (*8.31*), and

$$E_j = \text{const} + \tfrac{1}{2}\,\varepsilon\, j(j+1) \tag{4}$$

a symmetrical or neutral theory. In these formulae, $\varepsilon \equiv \hbar^2/I$, I being fined by (*1.35–16*); the exact relation of the symbol g to the source ıstants of the meson fields (*15.31*) depends on the types or mixtures 5.32) of fields assumed.

A3.12. *Scattering of mesons by nucleons.* It was first pointed out by EISENBERG [39] that the inertia of the spin and charge of a nucleon ıuld have a considerable effect in reducing the scattering cross-section mesons, especially for high meson energies, i.e. high frequencies of the son field. While ordinary perturbation theory, when this inertia effect neglected, leads to a cross-section increasing as the square of the meson ergy, the inclusion in the calculation of the proper field of the nucleon es rise to an extra factor which brings the cross-section down to a ite limiting value for very large meson energies. It must be stressed, wever, that even without regard to spin and charge inertia, a similar ɔdification of the scattering cross-section at high energies can be achieved an improvement of the perturbation theory, enabling it to take account the damping reaction of the meson field on the nucleon.

A3.13. *Electromagnetic properties of nucleons.* The proper meson field a nucleon gives rise to an anomalous magnetic moment, and also to a read of the charge distribution around the nucleon, leading to an ditional electrostatic self-energy. These effects have been carefully ıdied by HOURIET [45]. In first approximation, one finds that the omalous magnetic moments of proton and neutron would be equal and posite, while the mass difference between proton and neutron as a result the charge spread would have the wrong sign; and these defects cannot remedied by the correction terms appearing in higher approximation.

A3.2. Properties of the deuteron

A3.21. *Static interaction.* The strong coupling form of static interaction between two nucleons can be derived by applying the method of nonical transformation mentioned above (*A3.11*) to the Hamiltonian of pair of nucleons at rest a certain distance r apart, interacting with e meson fields: the interaction potential appears among the terms proırtional to g^2 (i.e. of second degree with respect to the various source nstants). On symmetrical meson theory, the result can be expressed

very simply with the help of the transformation matrix effecting the transition from body-fixed axes to axes fixed in space *; in our notation, the elements of this matrix are conveniently represented by a set of three vectors $\vec{\mathbf{e}}$ in isotopic space, such that

$$\vec{s} = \vec{\mathbf{e}}\mathbf{t} \quad , \quad \mathbf{t} = \vec{\mathbf{e}}\vec{s}. \tag{1}$$

It is then found that the operators $\tau^{(1)}\tau^{(2)}\vec{\sigma}^{(1)}\vec{\sigma}^{(2)}$ and $\tau^{(1)}\tau^{(2)}\mathfrak{D}^{(12)}$ must be replaced (apart from a numerical factor) by $\vec{\mathbf{e}}^{(1)}\vec{\mathbf{e}}^{(2)}$ and

$$(\vec{\mathbf{e}}^{(1)}\vec{x}_0)(\vec{\mathbf{e}}^{(2)}\vec{x}_0) - \tfrac{1}{3}\vec{\mathbf{e}}^{(1)}\vec{\mathbf{e}}^{(2)},$$

respectively. Physically, this entails an intimate interdependency of the spin and charge degrees of freedom, which would result in a much more complex mixing of states of different orbital momenta and multiplicity than on the weak coupling scheme. In fact, the operators $\vec{\mathbf{e}}^{(1)}$, $\vec{\mathbf{e}}^{(2)}$ being given by their representation with respect to the individual quantum numbers

$$j^{(1)}, m^{(1)}, n^{(1)} \quad ; \quad j^{(2)}, m^{(2)}, n^{(2)},$$

one has to go over to the representation in terms of the quantum numbers J, K, L, I; $j^{(1)}, j^{(2)}$ of the two-nucleon system, defined as follows: J, K are the quantum numbers of the *spin* angular momentum and isotopic variable, while L, I are those of the orbital and total angular momentum:

$$(\vec{s}^{(1)} + \vec{s}^{(2)})^2 = J(J+1), \qquad (\mathbf{t}^{(1)} + \mathbf{t}^{(2)})^2 = K(K+1), \qquad \text{etc.}; \tag{2}$$

one has

$$\begin{aligned} |j^{(1)} - j^{(2)}| &\leqslant J, K \leqslant j^{(1)} + j^{(2)} \\ |J - L| &\leqslant \ I \ \leqslant J + L. \end{aligned} \tag{3}$$

Since there is now an infinite sequence of J-values corresponding to the isobaric states $j^{(1)}$, $j^{(2)}$, a state with a given value of I will involve a mixture of an infinity of states with different L and J; e.g. the ^3S and ^1S states of the deuteron will actually be mixtures of the respective types

$$^3S + {}^3D + {}^7D + {}^7G + {}^{11}G + \ldots \quad \text{and} \quad {}^1S + {}^5D + {}^9G + \ldots.$$

A3.22. *The S states of the deuteron.* In order to discuss quantitatively the bearing of this situation on the account of the known properties of the deuteron, VILLARS [46] has transposed to the strong coupling case Rarita and Schwinger's schematic treatment of the distance dependence of the potential (*16.2*); i.e. he adopts for the distance dependence of the central and non-central parts of the potential, characterized by the operators in $\vec{\mathbf{e}}$ and $\vec{x}_0$ given above, wells of different depths and equal widths. The constant ε occurring in the expression (*A3.11*-4) for the excitation energies

* See, e.g. H. CASIMIR, *Rotation of a rigid body in quantum mechanics*, thesis Leiden 1931, p. 7—11, 44 sq., 50 sq.

of the isobaric states is a further parameter, whose value may be varied within certain limits. In fact, we may write

$$\varepsilon = \eta M_m, \qquad \eta \approx \frac{\hbar}{g^2} a\varkappa, \tag{4}$$

and the condition (*1.35*–17) for strong coupling yields

$$\eta \ll 1. \tag{5}$$

Villars determines, for chosen values of ε and the potential width D, the potential depths required to account for the binding energy and quadrupole

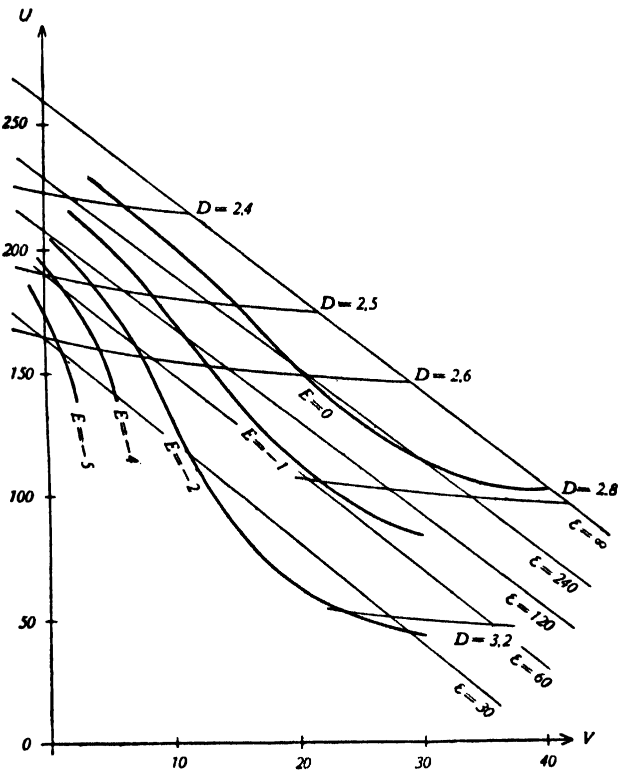

Fig. **A3.22.** Values of potential depths for different widths D and isobar excitation energies ε, according to VILLARS [46]. Villars' V and U correspond to $3J$ and $9\alpha J'$ respectively, in the notation of (*16.21*–1). The corresponding values of the energy E of the ^1S state of the deuteron are also indicated on the diagram. All energies are expressed in MeV, the width D in units 10^{-13} cm.

moment of the deuteron; for this purpose, he restricts himself to a mixture $^3S + {}^3D + {}^7D$ and verifies by control calculations that this approximation is permissible. He might have added the requirement of accounting for the magnetic moment, which would have fixed the width D for every assumed value of ε. From the figures he gives for the amounts of ^3D and ^7D admixtures, it would seem that, whatever the ε-value assumed, the

potential width should not be far from the value $D \approx d$ of Rarita and Schwinger's theory (which corresponds to $\varepsilon = \infty$); moreover, as we shall see presently (*A3.3*), the first saturation requirement implies that the distance-dependent factor V of the central potential operator $\vec{e}^{(1)}\,\vec{e}^{(2)}$ must be positive (corresponding to an attractive central potential in the ground state), and this condition puts a lower limit to the width, for which values not much below d are obtained: they vary from $2{,}6 \cdot 10^{-13}$ cm to $2{,}3 \cdot 10^{-13}$ cm for ε varying from 30 MeV to ∞. As regards the potential depths, that pertaining to the non-central part of the interaction hardly shows any dependence on the value of ε: just a slight increase with decreasing ε; the depth of the central potential, on the other hand, decreases markedly with decreasing ε: for $\varepsilon = 30$ MeV, e.g., it is less than half the Rarita-Schwinger value pertaining to $\varepsilon = \infty$. All these features are clearly illustrated by fig. *A3.22*, reproduced from Villars' paper.

Having fixed the potential depths, Villars proceeds to calculate the energy E of the ${}^1S + {}^5D$ state. The ${}^5D$ admixture is found to depress this state to a considerable extent, so that it generally becomes a state of binding unless very high values of ε are assumed. In fact, the accompanying diagram shows that a virtual level cannot be obtained for any ε below $\approx$ 200 MeV. Since this figure is far larger than the extreme limit of $\approx$ 100 MeV above which the condition (4), (5) for strong coupling is no longer fulfilled, the outcome of this investigation is to show rather strikingly that the actual situation is much nearer the weak than the strong coupling case.

A3.23. *Fast neutron scattering by protons.* Even when one disregards the effect of admixture on the ${}^3P$ states, one finds that the effective potentials in the ${}^3P_{0,1,2}$ substates have radically different properties according to the strength of the coupling, i.e. the value of ε. In comparison with the Rarita-Schwinger case, the effect of a finite value of ε is to depress the effective potentials of the three substates, with the result that both the ${}^3P_0$ and ${}^3P_2$ potentials become attractive and the scattering ratio Λ (*8.33*–23) is accordingly decreased; for $\varepsilon = 180$ MeV and 14 MeV neutrons, Villars finds $\Lambda = 0{,}97$.

A3.24. *Charge independence of nuclear interactions.* It might be thought that a strong coupling theory could give rise to charge-independent interactions even with charged meson fields only (*8.31*). But an explicit calculation by JOST [46] has established that the approximate equality of like and unlike particle interactions cannot be obtained on this basis.

A3.25. *Excitation of isobaric states.* Processes involving a sufficient amount of energy should be expected to lead to excitation of isobaric states of the participating nucleons. Some instructive estimates of the effects to be expected have been published. JAUCH [46] discusses the *inelastic photo-dissociation of the deuteron.* Assuming the first state of excitation of the

nucleon to lie at about 45 MeV, he finds that the ratio of the photoelectric cross-sections (the only ones of importance for high energies) for the inelastic and elastic disintegrations has a maximum of about 0,07, which is reached for an initial γ-ray energy of 87 MeV. LOPES [46, 47] has calculated, upon various assumptions about the energy of the first excited state, the *inelastic scattering cross-section of very fast neutrons by protons:* he finds for the ratio of this cross-section to that for elastic scattering values of the order 10^{-3} as soon as the excitation energy of the first isobaric state is of the order of a few hundred MeV.

A3.3. Saturation requirements

The problem of the saturation of the nuclear interactions has been studied on strong coupling theory especially by COESTER [44]. His treatment is based on the approximation of the individual model; further, he replaces the non-central part of the potential by its average over all angles, so that he deals essentially with an effective central potential of the general form $J(r)\,\mathcal{O}$, $\mathcal{O}$ being the operator which embodies the spin and isotopic variables: in a symmetrical theory, we have (*A3.21*) $\mathcal{O} = \vec{e}^{(1)}\,\vec{e}^{(2)}$; in a neutral theory, $\mathcal{O} \equiv \vec{e}_3^{(1)}\,\vec{e}_3^{(2)}$.

The *first saturation requirement* (*11.13*) is fulfilled by all theories for which $J(r) > 0$. Coester's proof of this general saturation condition may be illustrated by the simplest case of a neutral theory. In this case, the operator $\mathcal{O}$ is the cosine of the angle α between the two unit vectors $\vec{e}_3^{(1)}, \vec{e}_3^{(2)}$ symbolizing the spin states of the interacting nucleons. At small distances, i.e. $|J(r^{(12)})| \gg \varepsilon$, the lowest stationary state of the system of two nucleons is obtained by minimizing the potential energy $J(r)\cos\alpha$, i.e. for $\alpha = 0$ if $J < 0$ and for $\alpha = \pi$ if $J > 0$. For a system of A nucleons, the former possibility leads to a "freezing" of all vectors $\vec{e}_3^{(i)}$ parallel to each other and, consequently, to a predominance of ordinary non-saturation forces. If, however, $J > 0$, the vectors $\vec{e}_3^{(i)}$ will tend to "repel" each other and to distribute themselves uniformly over all directions: the resulting average of the ordinary part of the total potential energy therefore vanishes, at any rate in the approximation of the Fermi gas model.

The *second saturation requirement* presents a special problem owing to the existence of isobaric nuclei of arbitrary charge: it must be examined whether this might not lead to a greater stability of nuclei in which a certain number of neutrons or protons have been replaced by isobars of negative charge, thus leading to a smaller charge number than expected for a system of neutrons and protons. The problem is discussed by Coester in the Fermi gas model approximation; he treats the limiting cases of both large and small ε [i.e. ε large or small compared with the maximum kinetic energy E_m of a nucleon (*9.41–5*)], but finds it possible to interpolate

between them, so as to obtain a rough idea of the behaviour of the total energy over the whole range. Neglecting Coulomb energy, one finds that no stable nucleus will exist if the values of g and ε are too small; in fact, we must have, in the first place,

$$\frac{g^2}{\hbar} \approx 5 \frac{M_m}{M}; \tag{1}$$

nuclear radii of the right order of magnitude are then obtained for

$$\varepsilon \approx 10 \,\mathrm{MeV}. \tag{2}$$

The calculation of the Coulomb energy shows that condition (1) is also compatible with the empirical dependence of the charge number Z on the mass number A. However, the approximation of the Fermi gas model, on which these conclusions are based, is far from reliable.

ADDENDA ET CORRIGENDA

ADDENDA ET CORRIGENDA

Pitié pour nous qui combattons toujours aux frontières
De l'illimité et de l'avenir
Pitié pour nos erreurs pitié pour nos péchés
GUILLAUME APOLLINAIRE, *Calligrammes*

Besides correcting the mistakes that came to my notice, I have endeavoured, in the following, to summarize the contents of the latest publications relating to the various sections of the book. This was no easy task, as it had to include, among others, an account of the new discoveries of different kinds of mesons and of the bearing of these discoveries on our conceptions of a field theory of nuclear forces. The most important additions are distinguished by a mark ⋇; some of the additional matter I have found convenient to present in a *new subsection*, marked in the same way. Cross-references to any part of the *Addenda* will be preceded by the word "ad", e.g. (ad *1.332*). The greatest part of the publications covered by the *Addenda* are contained in vol. **72** and **73** of the *Physical Review* and vol. **160** and **161** of *Nature*.

1.132. *Existence of the neutrino.* A very interesting survey of the whole problem has been published by H. CRANE (*Rev. mod. Phys.* **20**, 278. 1948). The study, by B. WRIGHT (*P.R.* **71**, 839. 1947), of the recoil nuclei from the *K*-capture process of ^{107}Cd was somewhat inconclusive. More interesting are the experiments of R. CHRISTY *et al.* (*P.R.* **72**, 698. 1947) on the β-decay of ^{8}Li. The product nucleus ^{8}Be disintegrates into two α-particles, whose cloud chamber tracks may be studied : from the angle between those tracks and from their ranges the existence of a recoil momentum and an estimate of its magnitude can be inferred. The examination of 217 cases, including 28 in which the electron track was also visible, tended on the whole to favour the neutrino idea: no marked correlation was found between the directions of emission of the electron and the neutrino. Still more recently, C. SHERWIN has developed a much improved technique, based on the production of monoatomic layers of the β-emitter under investigation; this makes possible a more detailed quantitative study of the β-process. The electrons and recoil nuclei emitted in selected directions are suitably detected, and from the time of flight of the recoil nucleus its momentum is deduced. In this way, curves of distribution of recoil momenta are obtained for different values of the angle between the electron and the recoil nucleus. Experiments have been carried out with ^{32}P (*P.R.* **73**, 216. 1948) and ^{90}Y (*P.R.* **73**, 1173. 1948). In both cases the analysis of the results shows that there is no conservation of momentum for the electron and the product nucleus only. Sherwin further tries to derive from his data

a definite law of correlation between the directions of emission of electron and neutrino; but this part of his analysis is not very convincing.

※ **1.14.** *Anomalous magnetic moment of electron.* From precise comparisons of the atomic Landé factors pertaining to certain states of the Ga and Na atoms, P. KUSCH and H. FOLEY (*P.R.* **72**, 1256. 1947; **73**, 412. 1948) * drew the important conclusion that *the intrinsic magnetic moment of the electron differs from the Bohr magneton by a factor 1,0012.* This anomaly has found a remarkable theoretical interpretation through the new development of quantum electrodynamics initiated by the discovery of a small anomaly in the term values of the hydrogen spectrum. J. SCHWINGER (*P.R.* **73**, 416. 1948) pointed out that the electromagnetic radiation field produced by an electron in an external field should have a small finite effect on the coupling of the electron with this external field, and he showed that both the anomalies mentioned above were of this type. In particular, for the *radiative correction* $\delta\mu_e$ *to the intrinsic magnetic moment* μ_0 *of the electron* he derived the value

$$\delta\mu_e = \frac{\pi}{2} \cdot \frac{e^2}{\hbar} \cdot \mu_0 = 0{,}001162\,\mu_0,$$

in excellent agreement with Kusch and Foley's results.

A consequence of the existence of the magnetic radiative anomaly is that the values of those *nuclear magnetic moments* which were determined in relation to that of the electron should be increased by the same factor. In most cases, however, this correction falls within the margin of uncertainty of the measurements. Only for the very precisely determined magnetic moments of the proton, neutron and deuteron does it lead to significant changes. The figures collected in table *U.22* must be replaced (there and in the formulae quoted in the last column of the table) by the following:

$$\begin{aligned} \mu_p &= 2{,}7928 \\ \mu_n &= -1{,}9125 \\ \mu_d^0 &= 0{,}8803 \\ \mu_d &= 0{,}8575. \end{aligned}$$

In Appendix II, and especially in table *A2.21–2, account has been taken of the correction* so far as was thought necessary.

The small modification imposed by this correction upon the highly accurate values of μ_p and μ_d has removed a puzzling discrepancy that had arisen between the results of very precise measurements of the hyperfine doublet separations for hydrogen and deuterium and the values of these separations calculated by means of the uncorrected magnetic moments (J. NAFE and E. NELSON, *P.R.* **73**, 718. 1948 **). A still more exacting test

* More detailed account: *P.R.* **74**, 250. 1948. See further J. MCNALLY, *P.R.* **73**, 1130. 1948, who applied the same method to the Ca II spectrum, with a similar result.

** This paper contains all previous references.

is provided by the comparison of the empirical ratio of the hyperfine doublet separations with that deduced from the ratio of the magnetic moments, which has lately been determined much more accurately than either μ_p or μ_d (A. ROBERTS, *P.R.* **72**, 979. 1947; F. BLOCH, E. LEVINTHAL and M. PACKARD, *P.R.* **72**, 1125. 1947; F. BITTER *et al.*, *P.R.* **72**, 1271. 1947):

$$\mu_p : \mu_d = 3{,}257195 \pm 0{,}00002.$$

In order to obtain complete agreement, it is then necessary (A. BOHR, *P.R.* **73**, 1109. 1948) to take into consideration the asymmetry of the magnetic interaction of the electron with the proton and the neutron in the deuterium atom.

※ **1.33.** *New discoveries about mesons.* When this subsection was handed over to the printer, in the spring of 1947, the exhortation to "be prepared for the eventuality that other kinds of mesons ... may also play a part, perhaps even the main part, in bringing about nuclear interactions" was by no means intended as a rhetorical flourish. It was inspired by the uncertainty arising from the unexpected behaviour of negative mesons slowed down in light materials (*1.332*). Shortly afterwards it indeed became clear, as will be more fully explained below (ad *1.332*), that the mesons observed under usual conditions had only a very weak interaction with nucleons and could not be made responsible for the forces between them. At the same time, however, a succession of spectacular discoveries revealed an unsuspected wealth of particles with different masses, intermediate between those of electron and nucleon, which offered new possibilities of interpretation of the nuclear field.

The study of the tracks produced in photographic emulsion (*6.42*) by cosmic radiation at high altitudes led Powell and his collaborators * to the following general conclusions: in nuclear explosions produced locally by the impinging radiation, charged particles of mass roughly 300 m can be emitted, of which some, the *σ-mesons*, are observed to produce nuclear explosions, showing as "stars" in the emulsion (*12.30*), while others, the *π-mesons*, decay with an extremely short life-time into a somewhat lighter charged particle, the *μ-meson*, and another neutral particle (we shall call it a *μ^0-meson*) which must have a finite mass again somewhat smaller than that of the *μ-meson*. In fact, while absolute mass determinations from the emulsion tracks are difficult **, the ratio of the masses of π- and μ-meson has been fixed with some accuracy ***:

$$M_\pi : M_\mu = 1{,}65 \pm 0{,}15.$$

* C. LATTES, G. OCCHIALINI and C. POWELL, *Nature* **159**, 694. 1947 (with H. MUIRHEAD); **160**, 453, 486. 1947; G. OCCHIALINI and C. POWELL, *Nature* **162**, 168. 1948. See also F. FRANK, *Nature* **160**, 525. 1947.

** For σ-mesons, see S. LATTIMORE, *Nature* **161**, 518. 1948. For both π- and σ-mesons, an estimate (270 ± 40) m is quoted in G. OCCHIALINI and C. POWELL, *Nature* **161**, 551. 1948 (referred to as O & P).

*** Value quoted in O & P. The method is described by C. LATTES, G. OCCHIALINI and C. POWELL, *Proc. Phys. Soc.* **61**, 173. 1948.

The evidence is thus compatible with the assignment of a mass of about 200 m to the μ-meson (identified with that hitherto observed in cosmic radiation) and of $(330 \pm 30)\ m$ to the π-meson; the mass of the μ^0-meson is then estimated from the energy balance of the decay process at about 100 m. It is natural to regard the π- and σ-mesons as particles of the same kind, with a positive and a negative charge respectively: before it has time to decay, the negative σ-meson is captured by a nucleus and thus induces a nuclear explosion.

A brilliant confirmation of these results was obtained when the same phenomena could be reproduced at Berkeley with much greater intensity by bombarding targets of various materials with α-particles from the large cyclotron. By using the emulsion technique, it was found that when the energy of the α-particles exceeded 300 MeV, π- and σ-mesons were produced in nuclear collisions with rapidly increasing yield *. The application of magnetic deflection, besides establishing definitely the sign of the charge of the π- and σ-mesons, made possible a much more accurate determination of their mass; a value

$$M_\pi = (313 \pm 16)\ m$$

was found, from which one derives

$$M_\mu = 200\ m \quad , \quad M_{\mu^0} = 90\ m.$$

Obviously the π-mesons ** are strongly coupled to the nucleons and must essentially contribute to the nuclear interactions; their mass gives a quite acceptable value for the range of the forces. In fact, a mass value around 300 m corresponds more closely to the range derived from proton-proton scattering experiments (*7.13*) than would a mass of 200 m. It is true, on the other hand, that the lower mass value seems to be favoured by other evidence derived from the binding energies of light nuclei (*14.22*), but this is of very little weight indeed; evidence from the proton-neutron system (*8.33, 8.34*) is inconclusive, but certainly not incompatible with a mass value $\approx 300\ m$. Precisely how the nuclear forces are brought about by the π-mesons will primarily depend on their spin, about which we have no experimental indication whatever. Much the simpler course, for the time being, is to *assume that they have integral spin* (0 or 1). It is then possible to take over all the discussions bearing on "mesons" in the text of the book, except that of subsection *15.35*; the only change being in the mass value. In particular, the charge independence of the nuclear interactions will then require the existence of neutral π-mesons (*8.31*), which would presumably be highly unstable with multiple photon emission (*8.311*).

The assumption that π-mesons have integral spin does not commit us to any definite value for the spin of the μ- and μ^0-mesons. Indeed, there

* A preliminary acount of the Berkeley results is given in O & P.

** Henceforth, we shall call *π-mesons* all those of the kind to which both π- and σ-mesons belong, whether they be positively or negatively charged, or possibly neutral.

is nothing in the known properties of these mesons to give the preference either to an integral or to a half-integral spin; cosmic ray evidence (*1.333*) only excludes spin 1, but is compatible with either 0 or $\frac{1}{2}$. In the following, we shall treat the case of half-integral spin; but the whole discussion would not be materially altered if we started from the alternative assumption. We shall thus consider the μ- and μ^0-mesons as two states (of different charges and masses) of a new kind of particle of spin $\frac{1}{2}$, for which we shall use the name "μ-meson" (just as we use "leptons" for both electrons and neutrinos). In this case, the decay process of a π-meson into a pair of μ-mesons can be treated in much the same way as its decay into a pair of leptons (*15.35*); we have to introduce a coupling between the π-meson and the pair consisting of a μ- and a μ^0-meson, with a set of coupling constants whose order of magnitude we shall represent by $\hat{g}$. The life-time of the π-meson is given by formulae somewhat more complicated than (*15.35*–*25*) owing to the occurrence of the non-vanishing mass of the μ^0-meson; they have been established for a pseudoscalar and a vector π-meson by C. MARTY and J. PRENTKI (*C.R.* **226**, 787, 1948).

Now, the order of magnitude of the life-time of the π-mesons produced in the Berkeley experiments can be estimated from a consideration of the paths they describe between the moment of their emission from the target and that of their decay in the emulsion; the result is 10^{-8} sec. It is remarkable that this implies for the coupling constants $\hat{g}$ roughly the same order of magnitude as that which had to be assumed for the coupling constants $\breve{g}$ between mesons and leptons in order to explain the β-decay by a mechanism involving the meson field as an intermediary between nucleons and leptons (*1.322*). In view of this coincidence, it is tempting to imagine that the π-mesons are coupled to pairs of μ-mesons and to pairs of leptons in *exactly* the same way (in fact, as if μ-mesons and leptons were different states of a single species of elementary particles of half-integral spin). The mechanism just mentioned for the β-decay could then be maintained in the form discussed in *15.35*; of course, all considerations developed in that subsection in connexion with a meson life-time of about 2 μsec (which is *that of the μ-meson*) lose any significance and should be ignored. On the other hand, it is just as likely, for all we know *, that there is no direct coupling at all between π-mesons and leptons, and that the β-decay results from a direct coupling between nucleons and leptons, as assumed in Fermi's original theory, and symbolized, as regards the order of magnitude, by a "coupling constant" $g_F \approx g\breve{g}/\varkappa^2$. As to the decay of the μ-meson, it raises a problem by itself, which will be discussed below (ad *1.332*) in the light of new evidence.

According to the above scheme, there is an indirect coupling between

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* The decay of a π-meson into *leptons* would not be observable with the present emulsion technique.

a nucleon and a pair of μ-mesons, through the intermediary of a π-meson; on account of $\hat{g} \approx \breve{g}$, the corresponding coupling constant is of the same order of magnitude g_F as that between nucleons and leptons. Such a weak coupling is just what we should expect from the fact that a negative meson slowed down to the "*K*-orbit" around a light nucleus has time to decay before being captured by the nucleus; a straightforward estimate (A. LODGE, *Nature* **161**, 809. 1948) of the capture probability on the basis of the theory just outlined agrees even quantitatively with the latest data on negative meson decay (ad *1.332*). The smallness of the interaction between μ-mesons and nucleons is also in qualitative agreement with the great power of penetration of cosmic ray (μ-)mesons through matter, as shown by their small scattering cross section in air and the presence of an appreciable number of mesons even deep underground. In fact, the contrast between the large cross section (of the order of nuclear dimensions) for the production of the hard component of cosmic radiation by the primary protons and the 100 times smaller scattering cross section of mesons by nucleons had led to the hypothesis of an intermediate step in the process of μ-meson production even before the discovery of the π-meson (R. MARSHAK and H. BETHE, *P.R.* **72**, 506. 1947; V. WEISSKOPF, *P.R.* **72**, 510. 1947). It is therefore rather puzzling that in the Berkeley experiments μ- and π-mesons seem to be produced in comparable numbers in the target; the interpretation of the results, however, is not yet quite definite.

If, as the above scheme implies, the μ-mesons interact with nucleons only in pairs, the capture process of a charged μ-meson should be accompanied by the emission of a (neutral) μ^0-meson, carrying away at least 45 MeV energy. This mechanism would explain the empirical fact (cf. O. PICCIONI, *P.R.* **73**, 411. 1948; J. RETALLACK, *P.R.* **73**, 921. 1948) that the μ-meson capture does not give rise to nuclear excitation and explosion.

The π-mesons and properties of cosmic radiation. In view of the discovery of the π-meson, the usually accepted picture of the production of the (μ-)meson component of cosmic radiation has to be altered. The primary protons are expected to give rise, in nuclear collisions, to groups of charged and neutral π-mesons; and the charged component will then decay into charged and neutral μ-mesons. An interesting attempt has been made by G. CHEW (*P.R.* **73**, 1128. 1948) to analyse this mechanism in a more quantitative way. Starting from a schematized model of the multiple emission of π-mesons by excited nuclei, he fixes the parameters entering into the description of the model by comparison with suitably chosen data on the distribution of fast μ-mesons at different altitudes and latitudes; he eventually obtains a remarkably accurate check of his calculated meson distribution in energy at sea-level by the relevant empirical data. He estimates the mean free path of the primary protons for nuclear collisions at ≈ 5 cm Hg, i.e. somewhat more than would correspond to a cross-section of nuclear dimensions; the average multiplicity of π-meson production would be ≈ 5 at high altitude and would increase as the square root of

the proton energy. It is interesting that he finds that only about half the primary energy goes into the production of *charged* π-mesons.

The neutral π-mesons, supposed to be produced in the same nuclear process, would be expected to decay almost immediately into two or more photons, which would start cascade showers of the usual type (*8.311*). K. GREISEN (*P.R.* **73**, 521, 2nd letter. 1948) enumerates a whole series of features of cosmic radiation in qualitative support of this view: the increase of burst production with altitude; the large amount of soft component in giant air showers; certain cloud chamber evidence of electrons emitted in nuclear explosions (as consecutive products of the photon decay of neutral π-mesons); the comparison of the electron with the meson spectrum, revealing the double origin of the former (decay of neutral π-mesons and of charged μ-mesons). He further suggests various experiments capable of throwing light on this aspect of the question.

A rather close estimate of the life-time of the π-mesons can be derived by considering the variation of the number of (μ-)mesons with depth underground; in fact, this distribution reflects the conditions under which these mesons are produced near the top of the atmosphere by decay of π-mesons of very high energies. It can easily be shown (K. GREISEN, *P.R.* **73**, 521, 1st letter. 1948) that the occurrence of the π-meson as an intermediate step in the production of μ-mesons by nucleons has the effect of modifying the rate of production by a factor $\left[1 + \frac{R(p)}{z_0}\right]^{-1}$, z_0 being the height of the homogeneous atmosphere and $R(p)$ the range of a π-meson of momentum p, i.e. $R(p) = ct_0p/M_\pi$, where t_0 is the life-time of the π-meson at rest. From the change of slope in the distribution curve of the μ-mesons with depth, i.e. with the momentum p, one estimates $t_0 \approx 6 \cdot 10^{-8}$ sec. The correctness of this interpretation of the change of slope of the distribution curve is confirmed by the existence of a temperature effect.

1.331. *"Heavy" mesons.* Some additional evidence of "heavy" mesons with a mass around 1000 m has been presented: (*a*) G. ROCHESTER and C. BUTLER (*Nature* **160**, 855. 1947) interpret two events occurring in cloud chamber pictures of penetrating showers as evidence for the spontaneous decay of a neutral and a charged "heavy" meson into lighter mesons; (*b*) L. LEPRINCE-RINGUET *et al.* (*C.R.* **226**, 1897. 1948) have observed in a photographic emulsion an event that must probably be interpreted as the production of a star by a "heavy" meson. It is, of course, not known whether these various instances refer to a single kind of particles, and any speculation as to the place occupied by "heavy" mesons in the complicated constellation of "elementary" particles would be premature.

※ **1.332.** *Behaviour of negative mesons slowed down in light materials* *. The effect discovered by Conversi, Pancini and Piccioni has now been studied for the following materials:

$_4$Be	: SIGURGEIRSSON and YAMAKAWA [47], VALLEY [47]	$_{16}$S	: SIGURGEIRSSON and YAMAKAWA [47]
$_5$B	: NERESON [48]	$_{26}$Fe	: RASETTI [41], CONVERSI *et al.* [47], VALLEY [47], NERESON [48]
water	: VALLEY [47]	$_{29}$Cu	: VALLEY [47]
$_6$C	: CONVERSI *et al.* [47], VALLEY [47], NERESON [48]	$_{82}$Pb	: NERESON and ROSSI [42, 43].
$_{11}$Na$_9$F	: TICHO and SCHEIN [48]		
$_{13}$Al	: RASETTI [41], TICHO [47*a*], VALLEY and ROSSI [48], NERESON [48]		

In all elements from $_{16}$S onwards no decay of negative mesons has been observed; in particular, Nereson and Rossi's precise measurement of the life-time $t_0 = 2{,}15$ μsec refers to the *positive* mesons. By recording cloud chamber pictures of the decaying mesons, VALLEY [47] was able to establish that the mesons concerned are "μ-mesons" of mass 200 m.

It was discovered by TICHO and SCHEIN [47*a*, *b*, 48] and independently by VALLEY and ROSSI [48], that while the life-time of the positive mesons has the constant value t_0, that of the negative mesons exhibits a rapid decrease with increasing atomic number of the absorber: for NaF, the apparent life-time of the negative particles is $1{,}33 \pm 0{,}14$ μsec, for Al it is $0{,}74 \pm 0{,}17$ μsec. The natural interpretation of this result is that in the case of the negative mesons, there is competition between spontaneous decay with the life-time t_0 and capture by the nuclei of the absorber. The life-time t_c corresponding to the latter process will be expected to vary rapidly with the atomic number; simple considerations (J. WHEELER, *P.R.* **71**, 320. 1947) show that one would expect $1/t_c \sim Z^4$. The above figures for the apparent life-time $t_{\mathrm{app}} = \left(\frac{1}{t_0} + \frac{1}{t_c}\right)^{-1}$ are in fact compatible with the relation

$$\frac{t_0}{t_c} = \left(\frac{Z}{Z_0}\right)^4, \quad Z_0 \approx 11.$$

The fraction of decaying negative mesons can be measured directly and compared with the calculated value $t_{\;\mathrm{rp}}/t_0$; this check on the consistency of the proposed interpretation comes out satisfactorily in Ticho and Schein's

* The following recent papers will be listed and quoted here according to the system (*N.* 22) adopted throughout the book:

H. TICHO
1947*a*: — , *P.R.* **72**, 255.
1947*b*: — and M. SCHEIN, *P.R.* **72**, 248.
1948: — and M. SCHEIN, *P.R.* **73**, 81.

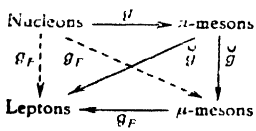
N. NERESON 1948, *P.R.* **73**, 565.

G. VALLEY
1947: — , *P.R.* **72**, 772.
1948: — and B. ROSSI, *P.R.* **73**, 177.

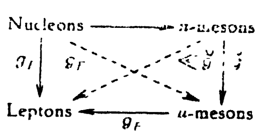
case, while there remains a discrepancy of uncertain origin in Valley and Rossi's measurements *.

From the theoretical side, unpublished calculations by FERRETTI and by BETHE have made it clear that the ingenious mechanism suggested by Bohr is not sufficient to prevent a negative meson from reaching the "K-orbit" in a time very short compared with the life-time for capture t_c indicated by the experiments. There is therefore no escape from the conclusion ** that the interaction of the μ-mesons with the nuclei is about 10^{12} times smaller than it would be if those mesons were responsible for the nuclear field.

Nature of μ-meson decay. New cloud chamber pictures of decaying μ-mesons, of which only one has hitherto been published (C. ANDERSON *et al.*, *P.R.* **72**, 724. 1947), make it quite clear that the balance of energy and momentum not accounted for by the observed decay electron is not simply carried away by a neutrino of small or vanishing mass. In fact, in Anderson's picture, the decay positon had an energy of only 24 MeV instead of the 50 MeV expected on the usual lepton pair hypothesis. Other pictures (if their interpretation is correct) seem to indicate varying energies of the decay electron between 21 ± 6 and 55 ± 15 MeV. This feature points to a process involving more than two particles. In view of the close association of μ- and μ^0-mesons in the decay of the π-meson, it is tempting to assume that besides an electron and a neutrino, the decaying μ-meson produces a neutral μ^0-meson; the so-called decay of the charged μ-meson might thus more appropriately be called a transmutation into a neutral μ^0-meson with emission of a lepton pair. Theoretically, this process would be described by introducing a direct coupling between the μ–μ^0-meson pair and the electron-neutrino pair; the coupling parameter, determined by the life-time t_0 of about 2 μsec, would be of the order of magnitude g_F. The complete net of interactions between the various kinds of particles might then be symbolized by either of the following schemes, in which dotted arrows represent indirect couplings:



or



The spin of the μ-mesons is left unspecified.

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* An alternative theory, according to which the μ-meson would not be captured by the nucleus, but its decay would be stimulated by electromagnetic fields (in particular by the Coulomb field of the nucleus), seems to be disproved by various observations. negative mesons stopped in a cloud chamber are not found to be associated with decay electrons (J. RETALLACK, *P.R.* **73**, 921. 1948); likewise, an analysis (O. PICCIONI, *P.R.* **73**, 411. 1948) of RASETTI's [41] experiments shows that not all negative mesons stopped in the absorber are accompanied by decay electrons: the meson decay with photon emission to be expected on such a view has not been observed (see below). On the theory of stimulated meson decay, see S. EPSTEIN, R. FINKELSTEIN and J. OPPENHEIMER (*P.R.* **73**, 1140. 1948).

** Cf. E. FERMI, E. TELLER and V. WEISKOPF, *P.R.* **71**, 314. 1947; E. FERMI and E. TELLER, *P.R.* **72**, 399. 1947.

A search for γ-rays that might accompany the decay process of the μ-mesons has been carried out by E. HINCKS and B. PONTECORVO (*P.R.* **73**, 257, 1122. 1948). They have established that the decay does not consist of the emission of an electron and a photon *.

2.1. *Nuclear radii.* From an improved theory of natural α-radioactivity, M. PRESTON (*P.R.* **71**, 865. 1947) derives a new estimate of the radii of heavy nuclei, somewhat larger than that given by (*2.1*–1, 2).

2.21. Attention is called to E. FEENBERG's detailed semi-empirical analysis of the nuclear energy surface (*Rev. mod. Phys.* **19**, 239. 1947). This should also be consulted in connexion with sections **3.22**, **3.4**, **10.3** and **12.1**.

3.41. On page 36, the last word of line 6 should be "low".

3.42. On page 38, first line, replace "smaller" by "smaller or larger".

5.121. In formula (*5.121*–26), the last term on the right hand side must have the *minus* sign.

5.31. *Calculation of phases.* L. HULTHÉN (*Arkiv* **35A**, No. 25. 1948) has further elaborated his variational method. Another method of approximation has been developed by W. RAMSEY (*Proc. Cambr.* **44**, 87. 1948).

6.122. It has been pointed out by M. HAMERMESH (*P.R.* **73**, 638. 1948) that there is an error in CASIMIR's [40] calculation of the effect of the quadrupole field of the deuteron in accelerating the para-ortho conversion of deuterium dissolved in water; the result given by Casimir is too large by a factor 5.

6.22. *Scattering of slow neutrons by protons.* The availability of intense sources of slow neutrons monochromatized by "velocity spectrometers" (cf. *6.31*) has greatly facilitated scattering cross-section determinations in the range of slow neutron velocities. The variation of the cross-section with velocity can now be studied continuously; in the thermal region the increase of the cross-section with decreasing neutron energy

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* An old argument of L. NORDHEIM (*P.R.* **59**, 554. 1941) may be revived in this connexion. Nordheim pointed out that the ratio of the intensity of the part of the soft component of cosmic radiation which arises from the decay of the hard (μ-meson) component, to the intensity of the latter can be calculated on cascade theory on the assumption that every meson gives a definite fraction α of its energy to cascade producing particles (electrons or photons): the comparison with experimental data would thus lead to an estimate of α. Assuming that the energy of the meson is equally shared by all its decay products, one would expect $\alpha = 1$ if the decay products are an electron and a photon, $\alpha = \frac{1}{2}$ if they are an electron and a neutral particle, $\alpha = \frac{1}{3}$ if they consist of an electron and two neutral particles, etc. The empirical evidence would seem to exclude definitely the value $\alpha = 1$ in favour of a smaller value; but it is not possible to decide, e.g., between $\frac{1}{2}$ or $\frac{1}{3}$.

owing to the binding of the scattering nuclei can be followed accurately and compared with theory. The cross-section for scattering by free nuclei is actually obtained with neutrons of a few eV energy. Scattering by paraffin has been studied in this way by L. RAINWATER *et al.* (*P.R.* **73**, 733. 1948).

The latest value of the proton scattering cross-section for neutrons of zero energy (*6.22*–32) is that obtained by E. MELKONIAN (*P.R.* **73**, 1265. 1948):

$$\mathcal{S} = (19{,}96 \pm 0{,}2) \cdot 10^{-24}\,\text{cm}^2.$$

It must be pointed out that the new measurements of the oxygen cross-section (E. MELKONIAN *et al.*, *P.R.* **73**, 1399. 1948) yield

$$\mathcal{S}_{\text{O}} = 3{,}68 \cdot 10^{-24}\,\text{cm}^2$$

instead of the higher value quoted in table *6.22*–3. This puts Simon's value of the proton cross section, in the same table, at $(14{,}6 \pm 0{,}8) \cdot 10^{-24}$ cm^2.

※ 6.31. *Scattering of slow neutrons by molecular hydrogen.* Alvarez and Pitzer's measurements have been extended, with improved technique, by R. SUTTON *et al.* (*P.R.* **72**, 1147. 1947). The scattering cross-sections of gaseous ortho- and parahydrogen at 19,5° K have been determined for the whole range of neutron temperatures 10° ... 30° K ($E = 0{,}8673$ ... 2,386 meV); the results are markedly different from the provisional values (*6.31*–1) obtained by Alvarez and Pitzer. For 20° K neutrons, e.g., Sutton *et al.* find

$$\begin{aligned} &\text{for } o\text{H}_2: && 124 \cdot 10^{-24}\,\text{cm}^2, \\ &\text{for } p\text{H}_2: && 3{,}97 \cdot 10^{-24}\,\text{cm}^2. \end{aligned}$$

※ 6.33. When subjected to the analysis outlined at the end of *6.33*, Sutton *et al.*'s results yield a value of the proton scattering cross-section for slow neutrons $\mathcal{S}(0) = 19{,}7 \cdot 10^{-24}$ cm^2, in excellent agreement with the direct measurements (*6.22*–32) of this quantity. Because of the smallness of the ratio $(3 \cdot {}^3a + {}^1a)^2/({}^3a - {}^1a)^2$, the quantity $\mathcal{S}(0)$ depends essentially on $\mathcal{S}_{\text{ortho}}$, as a comparison of (*6.33*–9) and (*6.32*–6) (in which $\mathcal{S}_{p\to o} = 0$) will show. The agreement just mentioned therefore confers additional reliability to the determination of $\mathcal{S}_{\text{ortho}}$. The measurement of $\mathcal{S}_{\text{para}}$, however, is more difficult and any error in its determination would mainly affect the value of the relatively small scattering radius ^3a. From the data, one finds

$${}^3a = 0{,}522 \cdot 10^{-12}\,\text{cm} \quad , \quad {}^1a = -2{,}34 \cdot 10^{-12}\,\text{cm}.$$

The singlet scattering radius ^1a is compatible (cf. table *6.431*–1) with a width $^1D \approx 2{,}8 \cdot 10^{-13}$ cm for the corresponding potential well, such as one would expect, assuming charge independence (*8.1*), from the proton-proton scattering data. But the triplet scattering radius ^3a yields a much smaller width value for the ^3S effective potential, viz.

$${}^3D = (1{,}5 \pm 0{,}4) \cdot 10^{-13}\,\text{cm}.$$

One would have to assume a one to two percent contamination of the parahydrogen sample with orthohydrogen to raise the triplet width derived from the experiments to the same value as ${}^1D$.

However, the low ${}^3D$ value has received a striking confirmation from quite independent experiments of E. WOLLAN and C. SHULL and their collaborators (*P.R.* **73**, 830, 842. 1948) on the diffraction of slow neutrons by crystalline powders. The distribution of the diffracted neutrons will depend on the *scattering amplitudes* characteristic of the nuclei of the atoms constituting the crystal; for hydrogen this scattering amplitude is clearly proportional to $(3 \cdot {}^3a + {}^1a)$. Now, by combining the results obtained with different suitably chosen crystals, it is possible to assign definite scattering amplitudes to the various nuclei involved. Thus, experiments with sodium containing crystals, including metallic Na and NaH, yield among others the hydrogen scattering amplitude. The resulting value of $(3 \cdot {}^3a + {}^1a)$ is in complete agreement with that derived from the molecular hydrogen data; combining it with the known value of $\mathcal{S}(0)$, one finally gets ${}^3D = (1{,}6 \pm 0{,}2) \cdot 10^{-13}$ cm.

There is of course no *a priori* objection to having effective potential wells of different widths for the ${}^3S$ and ${}^1S$ configurations of the proton-neutron system. It must be stressed, however, that a value of ${}^3D$ as low as that just derived would be difficult to reconcile with the electric quadrupole moment of the deuteron (see ad *16.21*).

6.35. J. SCHWINGER (*P.R.* **73**, 407. 1948) discusses the polarization of fast neutrons in their passage through matter, owing to the action of the nuclear Coulomb field. Indeed, the electrostatic potential V_e gives rise to a magnetic interaction (*15.22*–9) on the moving neutrons

$$\frac{\hbar}{2M^2}\mu_n \vec{\sigma}\,(\mathrm{grad}\, V_e \wedge \vec{p}).$$

6.41, 6.52. The most recent data on neutron scattering and capture cros-sections have been collected by H. GOLDSMITH, H. IBSER and B. FELD (*Rev. mod. Phys.* **19**, 259. 1947).

⁂ **6.413.** *Scattering of very high energy neutrons by protons.* When the 180 MeV deuterons from the Berkeley cyclotron strike a target, they produce a narrow beam of high energy neutrons as a result of grazing collisions with the nuclei of the target, in which the proton of the deuteron is stripped off, while the neutron continues on its way. This phenomenon provides us with a powerful beam of neutrons, of well-defined direction and average energy 90 MeV. The total scattering cross-section of various nuclei for these neutrons has been measured by L. COOK *et al.* (*P.R.* **72**, 1264. 1947). For hydrogen, the result is $(0{,}083 \pm 0{,}004) \cdot 10^{-24}$ cm^2.

⁂ **6.42.** The angular distribution of the 90 MeV neutrons scattered by paraffin has also been determined in the angular range $\vartheta = 70^\circ \ldots 170^\circ$

(J. HADLEY *et al.*, *P.R.* **73**, 1114. 1948). In this range, the intensity of the scattered beam is found to increase steadily with angle, with some indication of a minimum about $\vartheta = 80°$. The anisotropy parameter

$$A = \mathcal{S}(\pi)/\mathcal{S}(\tfrac{1}{2}\pi)$$

is of the order of magnitude of 3. Preliminary observations indicate that for smaller scattering angles, the variation of the scattered intensity is roughly symmetrical to that just described, with presumably another maximum at $\vartheta = 0$; it is not clear whether this maximum is greater or smaller than that in the backward direction. The consequences of this new evidence for the theory of the nuclear field will be discussed below (ad *8.33*, ad *16.22*).

6.431. On page 128, line 14 (not counting the table), replace B by BH.

6.52, see ad *6.41*.

6.53. *Deuteron disintegration by electron impact.* The statement about the angular dependence of the effect, following formulae (23) on p. 139, needs correction. The differential cross-section of the electric effect includes an additional term containing the angular factor $Y_2^0(\vartheta)$, which arises from the longitudinal part of the field $\vec{E}_0$. This term does not give any contribution to the *total* cross-section.

The discussion of the magnetic exchange effect by Lubański and Rosenfeld, mentioned at the end, was based on the erroneous expression for K_{magn} given by Pais (*8.34*). Actually, the magnetic exchange effect is quite negligible.

7.12. To the "second group" of experiments (p. 148) must be added those of J. DEARNBY, C. OXLEY and J. PERRY (*P.R.* **73**, 1290. 1948) with protons of 7 MeV. The scattered protons were collected on photographic plates, the point of entry in the plate determining the scattering angle. In a note added in proof to their paper, the authors announce that preliminary results of new experiments yield larger cross-sections than those they obtain; it is therefore premature to consider the conclusions of the theoretical discussion they give. A detailed account of WILSON *et al.*'s experiments with 14,5 MeV protons has now been published in *P.R.* **72**, 1131. 1947.

As regards the reliability of the experiments of the "first group" (p. 149), it may be noted that the results obtained by Herb *et al.* at 2,1 MeV proton energy have been confirmed by new measurements at 2,08 MeV with the same apparatus (TASCHEK *et al.* [47]). On the other hand, measurements by a similar method, recently announced (W. SLEATER *et al.*, *P.R.* **73**, 1241. 1948), with protons of 2,4 and 3,0 MeV, yield values of scattering intensities at 2,4 MeV larger than Herb's by 2,9 % (while a probable error of only 2 % is claimed).

In table *7.12* (p. 150), first column, line 6 from the bottom, read 1200 instead of 1220.

7.13. An interesting attempt to establish an analytical formula giving the nuclear potential in terms of the phases has been made by C. FRÖBERG (*Arkiv* **34A**, No 28. 1948). However, since the analysis is entirely based on Born's approximation, the scope of the final formula is still too limited for practical applications. A more general iteration procedure has been outlined by E. HYLLERAAS (*P.R.* **74**, 48. 1948).

G. BREIT, A. BROYLES and M. HULL (*P.R.* **73**, 869. 1948) discuss the sensitivity of proton-proton scattering experiments with respect to the distance dependence of the nuclear potential.

The paper by J. LUBANSKI and C. DE JAGER is now published in *Physica* **14**, 8. 1948.

7.131. *Discussion of 10 MeV data.* According to a note by L. FOLDY (*P.R.* **72**, 731. 1947), the theoretical curves of fig. *7.131* correspond to a ^3P potential well of the same depth as the ^1S potential, yielding, when repulsive, a ^3P phase of — 1,4°. If we take for the ^1S phase the *theoretical* value of 52,5°, the data, as pointed out by Peierls and Preston, would indicate for the ^3P phase the somewhat lower figure of about — 0,8°; this they try to account for on the assumptions stated in the text.

The 10 MeV results have also been discussed by RAMSEY * on the assumption of the meson type of potential. A ^1S phase of the order of magnitude just mentioned might be accounted for by the potential corresponding to a meson mass of $(270 \pm 30)\ m$. The most likely choice of the strength of the ^3P effective potential (as given by the "symmetrical" theory, *8.32*) then leads to a ^3P phase of the right sign, but of too large an absolute value.

7.14, 12.33. The study of the penetration of a fast nucleon into nuclear matter has now been extended to the case of meson interaction by Dr. ZWANIKKEN. The results are not sensitive enough to the range of the interaction, however, to yield useful information concerning the value of the meson mass.

7.2. On page 157, line 3, after "For white dwarfs", insert: "it may be applied to the boundary layer; in the degenerate core,".

⁂ **8.33.** *Neutron-proton scattering at very high energies.* The experiments on scattering of 90 MeV neutrons have been the occasion of an unprecedented outburst of theoretical calculations, which does little credit to the system of free enterprise in the realm of physics. Each author having his own pet nuclear potential and set of constants, the result is a mass of unconnected information — *rudis indigestaque moles* — from which some general qualitative features emerge, which might have been obtained with less trouble in a more systematic way. In this subsection we shall only discuss the case of a *central nuclear potential*; the influence of non-central and non-static forces will be treated below (ad *16.22*). The following table gives a survey of the scope of the various contributions:

* *Proc. R.S.* **A 194**, 228. 1948.

Distance dependence	Range ^3S (10^{-13} cm)	Range ^1S (10^{-13} cm)	Isotopic factor *	Neutron energy MeV	Angular distribution given	Reference
Well	2,8		n, c, s	20...80	for 80 MeV	CAMAC and BETHE [48]
			n, c, s	83	yes	MASSEY *et al.* [48*b*]
			n, s	100	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi):\mathcal{S}(0)$ for s	EISENSTEIN & ROHRLICH [48]
			s	100	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi)$	WU [48*a*,*b*]
	2,8	2,8	s	80	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi)$	WU [48*b*]
	2,0	2,8	s	80	id.	
	1,5	2,8	s	80	id.	
	1,8	2,8	n	100	no	EISENSTEIN & ROHRLICH [48]
Exponential	2		c, s	100	yes	BARKER [48]
	1		c, s	100	yes	
Gauss	1,94	1,94	n, c, s	100	yes	WU [48*a*]
	1,39	1,94	s	100	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi)$	WU [48*b*]
	1,04	1,94	s	100	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi)$	
Meson ** ($M_m = 326m$)	1,18		s	80	$\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi)$	CHEW and GOLDBERGER [48]

* n = neutral, c = charged, s = symmetrical (*15.21*).

** Other (not reliable) data in EISENSTEIN and ROHRLICH's [48] and WU's [48*a*, *b*] papers.

F. BARKER 1948, *Nature* **161**, 736.
M. CAMAC and H. BETHE 1948, *P.R.* **73**, 191.
G. CHEW and M. GOLDBERGER 1948, *P.R.* **73**, 1409.
J. EISENSTEIN and F. ROHRLICH 1948, *P.R.* **73**, 641 (Erratum *ibid.* 1411).
H. MASSEY, E. BURHOP and T. HU 1948*b*, *P.R.* **73**, 1403.
T. WU 1948*a*, *P.R.* **73**, 934.
1948*b*, *P.R.* **73**, 1132.

The method used in all cases, except in that of the Gauss potential, is a straightforward numerical calculation of the phases, generally up to $l = 4$; Born's approximation for the higher phases has not been found reliable. A second approximation to Born's formula has been developed by WU [48*a*] and applied by him to the case of a Gauss potential.

The *total cross section* is insensitive to the choice of the distance dependence of the potential and varies very little with neutron energy in the range 80 ... 100 MeV. It also depends little on the choice of the isotopic factor, being largest in a neutral theory and smallest in a symmetrical one. In any case it is of the right order of magnitude, but rather too large. The *angular distribution* generally * has a minimum in the neighbourhood of $\vartheta \approx \frac{1}{2}\pi$ and a maximum as well in the forward as in the backward direction. But in a neutral theory, the forward maximum is much larger than the

* The case of a neutral interaction of the Gauss type is exceptional in yielding a differential cross-section which decreases monotonically with increasing angle of scattering.

other, while the situation is reversed in a symmetrical or charged theory; in conformity with the "exchange" character of the interaction in these cases; there is no great difference between charged and symmetrical theory. It must be stressed that the presence in all cases of a "secondary" maximum in the theoretical distribution makes the interpretation of the experimental results more complicated; so long as the behaviour of the differential cross-section at small angles is not more quantitatively elucidated, it will be premature to conclude (as is sometimes hastily done) that these results establish the impossibility of a predominantly neutral form of nuclear interaction.

For a more rapid comparison of theory and experiment, the *anisotropy ratio* $A = \mathcal{S}(\pi) : \mathcal{S}(\frac{1}{2}\pi)$ is therefore not sufficient to characterize the distribution; the ratio $\mathcal{S}(\frac{1}{2}\pi) : \mathcal{S}(0)$ ought also to be taken into consideration. At present, however, we have only the empirical estimate $A \approx 3$ at our disposal. This compares with theoretical values of A of the following order of magnitude:

Distance dependence	Neutral	Symmetrical or charged
Well	≈ 10	≈ 100
Gauss	—	≈ 60
Meson ($M_m = 326\,m$)	—	6,8

Obviously, the assumption of a single range central potential is inadequate. In view of the recent indications (ad *6.33*) of a smaller range of the ^3S effective potential, the effect of decreasing this range has been tried by WU [48*b*] (for the well and Gauss potentials; see above table): the total cross-section is only slightly increased, but the anisotropy ratio A falls steeply to a value of the order 5 or 6 for the smallest range value considered. However, the assumption of a small triplet range gives rise to difficulties (ad *16.21*), and it is therefore gratifying that the non-central part of the ^3S interaction is found just as effective in reducing the value of A (ad *16.22*).

8.34. Table *8.34*–1 should be completed by the following one, corresponding to a meson mass $M_m = 296\,m$, nearer to that of the π-mesons (ad *1.33*):

	Φ_{el} 10^{-28} cm^2	Φ_{magn} 10^{-28} cm^2	Φ_{total} 10^{-28} cm^2	H_{bar}	H_{lab}
Neutral theory	10,50	3,19	13,7	0,168	0,190
Symmetrical theory	10,05	3,21	13,3	0,176	0,198

All these results have been obtained by Dr. HULTHÉN with the collaboration of Mr. HANSSON. It will be noticed that the transformation from

the barycentric to the laboratory system of reference has an appreciable effect on the intensity ratio H.

The total photo-disintegration cross-section (table *8.34–2*) has now been measured with greater accuracy:

Reference	$h\nu$ MeV	Φ_{total} 10^{-24} cm^2
R. WILSON, C. COLLIE and H. HALBAN, *Nature* **162**, 185. 1948	2,62	16,2 ± 1,0
B. RUSSELL *et al.*, *P.R.* **73**, 545. 1948	2,76	16,4

12.33, see ad *7.14*.

13.11. According to A. HEMMENDINGER (*P.R.* **73**, 806. 1948), the ground state of ^8Be has an energy larger than that of two α-particles by $(0{,}116 \pm 0{,}010)$ MeV.

13.211. More data about the excited levels of α-nuclei will be found in HORNYAK and LAURITSEN's [48] survey article.

14.12. The free deuteron scattering cross-section for slow neutrons is $(3{,}3 \pm 0{,}2) \cdot 10^{-24}$ cm^2 (value for 5 eV neutrons), according to L. RAINWATER *et al.* (*P.R.* **73**, 733. 1948) (ad *6.22*).

The total scattering cross-section of the deuteron for 90 MeV neutrons (L. COOK *et al.*, *P.R.* **72**, 1264. 1947; see ad *6.413*) is

$$(0{,}117 \pm 0{,}005) \cdot 10^{-24} \text{ cm}^2.$$

Theoretical calculations on the scattering of high energy (100 ... 200 MeV) neutrons by deuterons have been carried out by T. WU and J. ASHKIN (*P.R.* **73**, 986. 1948), using Born's approximation; insofar as the Coulomb force can be neglected, the results are also applicable to proton-deuteron scattering. They assume a Gauss type of nuclear potential with neutral, charged or symmetrical form of isotopic factor and discuss the angular distribution both for elastic and inelastic scattering. The total cross-sections for 100 MeV neutrons are of the right order of magnitude, larger than the experimental figure on neutral or charged theory, and smaller on symmetrical theory. For 200 MeV neutrons, they become smaller by a factor ≈ 2. In all cases, they are definitely smaller than the corresponding sums of neutron-neutron and neutron-proton cross-sections (ad *16.22*). The formulae for the case of a non-central potential are also given in the paper quoted, but without numerical discussion.

15.34. In table *15.34*, the expression given for the scalar field must be replaced by 0. The simplification of the expressions originally given in Rosenfeld's paper for the scalar and pseudovector cases is effected by a canonical transformation of the nucleon variables; see a forthcoming note by LE COUTEUR and ROSENFELD.

15.35. In contradiction to the statement following table *15.35*, Sakata's

value of the β-decay constant on pseudoscalar meson theory, given in that table, does not correspond to the most natural choice of the Hamiltonian. As pointed out by E. NELSON (*P.R.* **60**, 830, 1941), such a choice would lead to a formula involving the product

$$\left(f_1 - f_2 \frac{M_p + M_m}{M_m}\right)^2 \cdot \left(\check{f}_1 - \check{f}_2 \frac{m}{M_m}\right)^2.$$

Since this contains the same combination of constants as the meson life-time, the exceptional position of the pseudoscalar meson field, discussed at the end of *15.35*, would not normally arise. This correction, of course, has now only academic interest.

⋇ **16.20, 16.21.** The question of how far the assumptions of the Rarita-Schwinger theory, especially that of equal ranges for the central and non-central forces, involve an arbitrary restriction, has been investigated at my request by Mr. DEMEUR. For this purpose, he introduced different ranges for the two kinds of potentials just mentioned and tried to adjust all the parameters of the theory thus extended so as to account for the properties of the ground state of the deuteron. His conclusion is that Rarita and Schwinger's original choice is essentially unique; of course, some slight variations (including unequal ranges) are allowed by the margin of uncertainty of the empirical data, the most sensitive one being the electric quadrupole moment. In particular, such a low value of the range of the central part of the ground state potential as seems to be indicated by the slow neutron interference effects (ad *6.33*) would be entirely excluded by the accepted value of the quadrupole moment *.

⋇ **16.22.** *Scattering of high energy neutrons by protons on Rarita and Schwinger's theory.* We have seen (ad *8.33*) that (as might be expected) the assumption of a central nuclear potential is unsuited to account for the observed angular distribution of the scattered neutrons of 90 MeV energy: it yields for the anisotropy ratio a value which is in any case too high and for most types of distance dependence much too high. It is therefore interesting that Rarita and Schwinger's theory, which seems to give a fairly accurate model of the static nuclear forces, including their axial dipole component, shows in this respect a definite trend in the right direction. Considering that a central meson potential gives a value of A only about twice too large, it may be conjectured that the inclusion of non-central terms in such a meson theory might achieve agreement with experiment. Leaving this, however, for future examination, I shall only collect here the data obtained on the simple Rarita-Schwinger theory. In this case, they

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* This conclusion was actually drawn already by Schwinger in 1941 by a more qualitative argument; see formula (*6.12*–13).

An investigation similar to Demeur's has just been published by W. GUINDON (*P.R.* **74**, 145, 1948).

Reference	Neutron energy MeV	Total cross section n 10^{-24} cm²	c 10^{-24} cm²	s 10^{-24} cm²	Anisotropy ratio n	c	s
MASSEY *et al.* [48*b*]	83	0,234*	0,158*	0,141*	0,58*	3,80*	9,56*
EISENSTEIN and ROHRLICH [48]	100	0,205	—	0,113	—	—	†
J. ASHKIN and T. WU, *P.R.* 73, 973. 1948	100	0,205	0,205	0,129*	0,31	3,69	10*
	150	0,143	0,143	0,114	0,011	5,43	17,6
	200	0,108	0,108	0,090*	0,030	12,1	35,5

† $\mathcal{S}(\pi):\mathcal{S}(\frac{1}{2}\pi):\mathcal{S}(0) = 4{,}03:0{,}51:1{,}22$

are fortunately coherent, since there was hardly anything else to do but to adopt the parameter values given in Rarita and Schwinger's paper. In the table above the references by authors' names are to the papers quoted in the table ad *8.33*; the asterisks distinguish the figures derived by exact calculation, the others having been obtained by Born's approximation.

Eisenstein and Rohrlich point out that decreasing the range of the *non-central* part of the potential has the effect of increasing the total cross-section and decreasing the anisotropy ratio. Both Massey *et al.* and Ashkin and Wu give complete angular distributions; the latter authors give them also for proton-proton scattering (neglecting the Coulomb force).

H. SNYDER and R. MARSHAK (*P.R.* **72**, 1253. 1947) have drawn attention to the fact that at such high energies as are dealt with here, the velocity dependent nuclear interactions might begin to become significant; at any rate, no precise comparison of theoretical and experimental results should be attempted without taking them into consideration. An estimate of the scattering cross-section including the effect of the non-static forces may be obtained by applying Møller's relativistic treatment to the meson field description of the interaction between nucleons: this corresponds to a relativistic extension of Born's method. In a special example, the authors quoted show that the result, for the energies considered, may differ by as much as a factor 2 from that of Born's approximation applied to the static interaction.

16.33. A detailed treatment of the ground state of the deuteron on Hulthén's unsymmetrical theory is given by C. FRÖBERG (*Arkiv* **35A**, No 17. 1948).

17.51. Addition to table *17.51*: D. ZAFFARANS, B. KERN and A. MITCHELL (*P.R.* **74**, 105. 1948) have measured the γ-ray from the reaction

$$^{6}\mathrm{Li}\ (d,n)\ ^{7}\mathrm{Be} \rightarrow {}^{7}\mathrm{Li},$$

using 11,5 MeV deuterons. They find for the excitation energy of ^{7}Li the value 0,474 ± 0,004 MeV.

A1.2. The large table of nuclei contains some additional data and references pertaining to the class (*b*) nuclei (table *A1.22*) ^{6}He, ^{14}C, ^{22}Na, ^{34}Cl, ^{38}K, ^{58}Cu. The fact that the decay of ^{22}Na leads to an excited state of the product nucleus might perhaps provide an explanation of the anomalous character of the process in this case.

A2.21. *New determinations of magnetic moments.* R. POUND (*P.R.* **73**, 112. 1948) gives the following values:

Nucleus	J	μ
$^{31}_{15}$P	$\frac{1}{2}$	(+)1,1314
$^{69}_{31}$Ga	$\frac{3}{2}$	2,0165
$^{71}_{31}$Ga	$\frac{3}{2}$	2,5611
$^{127}_{53}$I	$\frac{5}{2}$	2.8122

A2.31. Correction to table *A2.31*: according to new measurements of W. GORDY *et al.* (*P.R.* **74**, 243. 1948), the ratio of the quadrupole moments of ^{81}Br and ^{79}Br is 0,835.

Trivial misprints

2.21. P. 24, last line before the table, instead of "accuary" read "accuracy".

5.12. P. 70, line 3, on the right-hand side of the second formula (20), the argument of the cosine should be $z - \frac{1}{2}\, l\pi$.

6.21. P. 99, the left-hand side of formula (*6.21*–4) must be e^{ikz} (not e^{ikr}).

6.412. P. 121, table *6.412*, in the heading of the last column, instead of "cm" read "cm^2".

7.12. In the caption of fig. *7.12*–1, instead of "HEYDENBERG" read "HEYDENBURG". In the caption of fig. *7.12*–2, lines 6 and 7, instead of "according" read "according to".

7.131. P. 153, line 8 from bottom, the last word should read "exemplified".

8.1. P. 159, last line but one, instead of "on an arbitrary way" read "in an arbitrary way".

14.21. P. 302, formula (*14.21*–3), the binding energy of ^{5}He should be $-27{,}4$ MeV.

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Abbreviations

Arkiv	Arkiv för Matematik, Astronomi och Fysik
Cambr. Conf.	Cambridge International Conference on Elementary Particles (Volume I of the Report published by the Physical Society, London, 1947)
C.R.	Comptes Rendus des séances de l'Académie des Sciences de Paris
P.R.	Physical Review
Proc. Cambr.	Proceedings of the Cambridge Philosophical Society
Proc. Cop.	Det Kgl. Danske Videnskabernes Selskabs mat.-fysiske Meddelelser
Proc. Dublin	Proceedings of the Royal Irish Academy
Proc. Jap.	Proceedings of the Physico-Mathematical Society of Japan
Proc. Lund	Det Kgl. Fysiografiska Sällskapet i Lunds Förhandlingar
Proc. R.S.	Proceedings of the Royal Society, London
Sow. Phys.	Physikalische Zeitschrift der Sowjet-Union

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TABLE OF ATOMIC NUCLEI

INTRODUCTION

Description of table

In the following table, the nuclei are classified by *mass number*, which is, of course, the rational arrangement from a theoretical point of view. Only those properties which are directly related to nuclear forces are listed in the table; fig. *3.42* at the end of the book provides some information about the abundances of the stable isotopes, while the electromagnetic properties are fully dealt with in Appendix II.

Columns 1 to 5 give the characteristic numbers concerning the constituent nucleons and the chemical symbols. Use is made of the following types:

clarendon	for stable nuclei,
antique	for α-unstable nuclei *,
italics	for β-unstable nuclei.

Isomers are distinguished by an asterisk affecting the chemical symbol. Only those nuclei have been listed whose charge number is certain and mass number either certain or probable. Remaining uncertainties in this respect, or in the position of isomers, are indicated by brackets.

Columns 6, 7, 8 contain the characteristics of the decay of unstable nuclei. *Column 6* gives the life-time: s = second, m = minute, h = hour, d = day, a = year. *Column 7* describes the mode of decay: K = K-capture, $I.T.$ = isomeric transition. Whether or not the emission of γ-rays accompanies the decay process is indicated between brackets after the symbol of the mode of decay; when known, the different γ-rays emitted are enumerated and the relative probabilities of the various competing modes of decay given. The symbols for the γ-rays are provided with indices only when the decay scheme has been investigated, so that it is established that these γ-rays are emitted after the decay process; the order in which they appear between the brackets is that of their successive emission, in so far as this order can be fixed unambiguously **. *Column 8* contains the energies of the particles or γ-rays emitted. For α-rays it gives the total energy liberated (i.e. the energy of the α-particles plus that of the recoil nucleus); only the α-rays corresponding to transitions between ground levels are considered. For β-particles, the maximum energy of the spectrum is given. In each case (except for isomeric transitions), the γ-ray energies are listed under the product nucleus, since they correspond to transitions between levels of that nucleus.

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* For nuclei which are both α- and β-unstable, italics are used.

** E.g. in a case like ${}^{198}_{79}$Au, the order of succession of $\gamma_2\,\gamma_3$ cannot be fixed by means of the available evidence.

Columns 9 to 12 are concerned with binding energies; first the difference between atomic weight and mass number is given, then the total binding energy in two different units, finally the average binding energy per nucleon. In these data the last decimal has been retained only because it may become significant in the computation of *differences* of binding energies, which are often known with greater accuracy than the binding energies themselves.

List of references

The table is entirely the work of Mr. A. H. WAPSTRA * and presents, it is hoped, various improvements on the similar one of Mattauch and Flügge, on which it is primarily based. It embodies all "class *A*" and "class *B*" nuclei of G. SEABORG's table (*Rev. mod. Phys.* **16**, 1. 1944), those contained in the Plutonium project table (*Rev. mod. Phys.* **18**, 513. 1946) ** and later discovered or confirmed nuclei. Older references to the original publications will be found appended to the tables just quoted, and a similar list of more recent publications accompanies the present one.

This list, however, offers only a *selection* of the latest articles; it is hoped that, by consulting those, the reader will be able to trace any other reference he may be interested in. Usually, the articles quoted are those from which the data adopted in the table have been taken; whenever the quotation refers only to the data of some particular columns of the table, these columns are indicated by their ordinal numbers. In some cases, however, it has also been thought advisable to quote papers whose results show some discrepancy from those adopted; such quotations are preceded by the indication "disc". Finally, a number of items in the list are just explanatory remarks, indicating e.g. how a particular mass-defect given in the table has been computed. It must be noted that new data have been incorporated in the table as late as September 1948; this entails, unfortunately, some discrepancies of minor importance between the information given by the table and that contained either in the text or in the diagrams at the end of the book.

Additional remarks

Mass-defects of light nuclei ($A < 65$). The mass-defect determinations in this part of the table follow the general lines indicated by Mattauch and Flügge; the only modifications of M & F's figures are those rendered necessary by the results of recent investigations, as indicated in the list of references.

An independent study of the available evidence for the range $A = 20 \dots 38$ has also been carried out by H. R. ALLAN, who has very kindly put the results of his discussion at our disposal and carefully

* Now at the *Instituut voor Kernphysisch Onderzoek*, Ooster Ringdijk 18, Amsterdam.
** An article by J. SURUGUE (*J. Physique* 8, 145, 1946) has also been consulted.

compared them with the corresponding portion of the present table. This comparison reveals a satisfactory agreement up to $A = 28$, and again in the interval $A = 35 \dots 38$; in almost all cases, the differences between the two sets of mass-defects fall within the limits of the estimated accuracy of the determination. In the interval $A = 29 \dots 34$, however, there is a large unexplained discrepancy (of the order of one mMU) between the values derived from nuclear reaction data (adopted by Allan) and those based on the mass spectograph measurement of ^{32}S (adopted by M & F and in the present table).

The mass-defects of the nuclei in the interval $A = 39 \dots 43$ can only be fixed in relation to each other, but cannot yet be connected to the rest of the series; this is indicated, as in M & F, by the undetermined correction $-\varepsilon$.

Mass-defects of heavier nuclei ($A = 65 \dots 205$). For this group, M & F give only the mean packing fraction for all stable isotopes of the same element, which is of little use. The corrected table M & F * gives the individual mass-defects only of $^{203,\,205}Tl$, and of a few other nuclei which have a single stable isotope. In other cases, however, when the mean packing fraction is obtained for pairs of isotopes of nearly equal masses, this value can be attributed to each of them and the individual mass-defects estimated in this way. This has been done, using M & F*'s data, for such pairs as $^{69,\,71}Ga$, $^{99,\,101}Ru$, etc. Moreover, mass spectrographic data (quoted in M & F and M & F *) for other groups of isotopes, such as $^{78,\,82,\,84,\,86}Kr$, were used to estimate the corresponding mass-defects.

Pairs of neighbouring "stable" isobars. The three pairs of isobars $^{113}_{48}Cd$—$^{113}_{49}In$, $^{115}_{49}In$—$^{115}_{50}Sn$ and $^{123}_{51}Sb$—$^{123}_{52}Te$ are given as stable in violation of the general stability conditions (*3.42*). The newly discovered isotope $^{138}_{57}La$, if stable, would give rise to a double anomaly of the same kind, $^{138}_{56}Ba$—$^{138}_{57}La$—$^{138}_{58}Ce$. It is probable, however, that in those cases one of the partners of each pair is actually unstable, with a very long life-time, and transforms into the other one, by β-decay if the isobar of lower charge number is the unstable one, and by K-capture in the opposite case. Although all attempts to establish the occurrence of either transmutation have hitherto failed in the cases mentioned, the instance of the isobaric pair $^{187}_{75}Re$—$^{187}_{76}Os$ would tend to strengthen the belief in the correctness of the above conjecture. While this pair was until quite recently considered as another example of the anomaly under discussion (and is still indicated as such in fig. *3.42* and in the diagram of stable and unstable nuclei), it has now been found that $^{187}_{75}Re$ is β-active, with a life-time of $(4 \pm 1) \cdot 10^{12}$ years *.

Mass-defects of the heavy radioactive nuclei. For this class of nuclei, the data given by M & F * are erroneous (A. WAPSTRA, *Nature* **161**, 529. 1948). In order to obtain a more consistent set of data, Wapstra

* S. NALDRETT and W. LIBBY, *P.R.* **73**, 487 (Erratum 929). 1948; N. SUGARMAN and H. RICHTER, *P.R.* **73**, 1411. 1948.

first determines the differences between the mass-defects of the members of each radioactive family and that of its last member (viz. $^{206, 207, 208}Pb$ and ^{209}Bi); then, he fixes the relative positions of the different families on the energy surface by requiring that all isotopes of any element should lie on a set of parallel parabolas (one for each sheet of the energy surface); or, at any rate, that they should lie as closely as possible to such a set. The parameters of these parabolas can be fixed by using isotopes within the same family. The distance between the sheets of the energy surface corresponding to even and odd nuclei is found by considering the isobars of mass 210; the sheet corresponding to odd mass nuclei is then *assumed* to lie midway between the two others, an assumption supported *a posteriori* by the consistency of the results. In order to check this consistency, it is useful to construct a relief diagram of the type represented in fig. *3.41* (at the end of the book). It is interesting to note that the first diagram of this kind drawn up by Wapstra on the same assumptions as M & F about the decay schemes of the various nuclei exhibited at some places a "ruggedness" contrasting with the generally smooth appearance of the energy surface. The occurrence of such irregularities may point to mistakes in the interpretation of some decay scheme, or to some still unknown feature of a decay process. For instance, it was recognized by Wapstra that a much better mutual agreement could be achieved by assuming that the 0,970 MeV γ-ray observed to accompany the β-decay of ^{228}Ac follows the emission of the electron; moreover, the new mass-defect obtained for ^{225}Ra from an improved determination of the starting point for the Np family suggests that the β-decay of this nucleus is followed by the emission of a γ-ray (not yet observed) of about 1 MeV. The adjustment finally adopted by Wapstra is estimated by him to be accurate to about 0,5 MeV.

The latest discoveries of new radioactive series in the U, Th and Ac families may lead to further minor improvements of the mass-defects listed in the table; in particular, the new data might help to fix the position of the sheet for odd mass nuclei more definitely.

Z	Sym-bol	n	N	Half-life	Decay	Energy of radiation	W—A	|ℰ|		|ℰ
						MeV	10^{-4} MU	10^{-4} MU	MeV	M
2	3	4	5	6	7	8	9	10	11	1
0	*n*	1	1	30 *m*	β^-		89,45 ± 0,25	0,0	0,00	0
1	H	−1	0				81,31 ± 0,03	0,0	0,00	0,
1	H	0	1				147,24 ± 0,06	23,5	2,19	1,
1	*H*	1	2	12,1 ± 0,5 *a*	β^-	0,011 ± 0,002	170,04 ± 0,20	90,2	8,39	2,
2	He	−1	1				169,88 ± 0,20	82,2	7,65	2,
2	He	0	2				38,60 ± 0,31	302,9	28,20	7,
2	He	1	3	$\approx 6 . 10^{-20}$ *s*	$\alpha + n$		137	294	27,4	5,
2	*He*	2	4	0,85 ± 0,05 s	β^-	3,5 ± 0,6	207,1 ± 7	313,5	29,19	4,
3	Li	0	3				169,17 ± 0,51	343,1	31,94	5,
3	Li	1	4		γ	0,474 ± 0,004	181,63 ± 0,57	420,1	39,11	5,
4	*Be*	−1	3	43 *d*	*K* (γ)		190,89 ± 0,66	402,7	37,49	5,
3	*Li*	2	5	0,89 ± 0,02 *s*	β^-	12	249,67 ± 0,66	441,5	41,10	5,
4	Be	0	4	very short	2α	0,116 ± 0,010	78,07 ± 0,39	605,0	56,31	7,0
4	Be	1	5				149,58 ± 0,62	622,9	57,98	6,4
5	B	-1	4	very short	$2\alpha + p$		161,04 ± 0,68	603,3	56,16	6,2
4	*Be*	2	6	2,5 . 10^6 *a*	β^- (no γ)	0,560 ± 0,010	166,22 ± 1,36	695,7	64,76	6,4
5	B	0	5				161,69 ± 0,70	692,1	64,43	6,4
6	*C*	−2	4	8,8 ± 0,8 *s*	β^+	3,36 ± 0,10	208,6 ± 1,2	637,1	59,30	5,9
5	B	1	6				129,01 ± 0,50	814,2	75,80	6,7
6	*C*	−1	5	20,5 *m*	β^+	0,981 ± 0,005	150,17 ± 0,75	784,9	73,07	6,6
5	*B*	2	7	0,022 ± 0,002 *s*	β^-	12	168	864,7	80,49	6,7
6	C	0	6				38,80 ± 0,25	985,8	91,76	7,5
6	C	1	7				75,61 ± 0,43	1038,4	96,66	7,4
7	*N*	−1	6	9,93 ± 0,03 *m*	β^+ (no γ)	1,21 ± 0,005	99,04 ± 0,44	1006,8	93,72	7,2
6	*C*	2	8	5100 ± 200 *a*	β^- (no γ)	0,1563	77,41 ± 0,43	1126,1	104,82	7,4
7	N	0	7				75,30 ± 0,16	1120,0	104,26	7,4
7	N	1	8				48,70 ± 0,72	1236,1	115,06	7,6
8	*O*	−1	7	126 ± 5 *s*	β^+	1,7	78	1198,6	111,58	7,4
7	*N*	2	9	7,35 ± 0,05 *s*	20 % β^-	10,3 ± 0,3	110 ± 4	1263	117,6	7,3
					40 % β^- (γ_1)	4,3				
					40 % β^- (γ_2)	3,8				
8	O	0	8		γ_1	6,2 ± 0,1	0,00	1366,1	127,16	7,9
					γ_2	6,7 ± 0,3				
7	*N*	3	10	4,14 ± 0,04 *s*	β^- (*n*)	2,7 ± 0,4	119	1345	125,2	7,3
8	O	1	9				45,0 ± 0,6	1410,5	131,30	7,7
9	*F*	−1	8	70 ± 5 *s*	β^+	2,1	75,8	1371,6	127,68	7,51
8	O	2	10				48,5 ± 1,8	1496,5	139,30	7,7
9	*F*	0	9	112 ± 4 *m*	β^+	0,72 ± 0,02	67,0 ± 1,9	1469,8	136,82	7,60

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	$W-A$	$\|\mathcal{E}\|$		$\|\mathcal{E}\|/A$
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
19	8	O	3	11	$27 \pm 0{,}5$ s	30% β^-	$4{,}5 \pm 0{,}3$	$93{,}7 \pm 3{,}5$	1540,7	143,45	7,550
						70% $\beta^-(\gamma?)$	$2{,}9 \pm 0{,}3$				
	9	**F**	**1**	**10**				$45{,}4 \pm 1{,}2$	1580,9	147,16	7,745
	10	*Ne*	−1	9	$20{,}3 \pm 0{,}5$ s	β^+	2,20	79,8	1538,4	143,20	7,537
20	9	*F*	2	11	12 s	$\beta^-(\gamma)$	5,0	65,4	1650,3	153,63	7,682
	10	**Ne**	**0**	**10**		γ	2,2	$-11{,}05 \pm 0{,}61$	1718,7	159,98	7,999
21	**10**	**Ne**	**1**	**11**				$7{,}4 \pm 1{,}9$	1789,7	166,60	7,933
	11	*Na*	−1	10	23 ± 2 s	β^+		35	1754	163,3	7,775
22	**10**	**Ne**	**2**	**12**		γ_1	$1{,}30 \pm 0{,}03$	$-14{,}2 \pm 3{,}0$	1900,7	176,93	8,042
	11	*Na*	0	11	$3{,}0 \pm 0{,}2$ a	$\beta^+(\gamma_1)$	0,575	$17{,}0 \pm 3{,}1$	1861,4	173,30	7,877
23	10	*Ne*	3	13	$40{,}7 \pm 0{,}8$ s	β^-	$4{,}1 \pm 0{,}3$	$16{,}7 \pm 4{,}8$	1959,2	182,53	7,932
	11	**Na**	**1**	**12**				$-28{,}5 \pm 2{,}0$	1996,3	185,83	8,080
	12	*Mg*	−1	11	$11{,}8 \pm 0{,}3$ s	β^+	2,82	$12{,}7 \pm 3{,}9$	1947,0	181,24	7,880
24	11	*Na*	2	13	14,8 h	$\beta^-(\gamma_1\gamma_2)$	$1{,}390 \pm 0{,}005$	$-19{,}0 \pm 3{,}2$	2076,3	193,32	8,055
	12	**Mg**	**0**	**12**		γ_1	1,380	$-78{,}9 \pm 3{,}1$	2128,0	198,09	8,254
						γ_2	2,758				
25	11	*Na*	3	14	$58{,}2 \pm 1{,}3$ s	55% β^-	$3{,}7 \pm 0{,}3$	$-22{,}9 \pm 3{,}7$	2169,6	201,96	8,078
						45% $\beta^-(\gamma?)$	2,7				
	12	**Mg**	**1**	**13**				$-62{,}7 \pm 1{,}8$	2201,3	204,91	8,196
	13	*Al*	(−1)	(12)	≈ 8 s	β^+					
26	**12**	**Mg**	**2**	**14**				$-94{,}0 \pm 2{,}4$	2322,0	216,15	8,313
	13	*Al*	0	13	$7{,}25 \pm 0{,}2$ s	β^+	2,99	$-51{,}0 \pm 4$	2270,9	211,39	8,130
27	12	*Mg*	3	15	$10{,}1 \pm 0{,}1$ m	$\beta^-(\gamma_1)$	$1{,}75 \pm 0{,}05$	$-71{,}9 \pm 1{,}3$	2389,4	222,42	8,238
						25% $\beta^-(\gamma_2\gamma_3)$?					
	13	**Al**	**1**	**14**		γ_1	$1{,}05 \pm 0{,}08$	$-102{,}0 \pm 0{,}8$	2411,3	224,46	8,313
						γ_2	$0{,}84 \pm 0{,}03$				
						γ_3	$0{,}64 \pm 0{,}03$				
	14	*Si*	−1	13	4,92 s	β^+	3,54	$-52{,}0 \pm 1{,}8$	2353,2	219,05	8,113
28	13	*Al*	2	15	$2{,}30 \pm 0{,}03$ m	$\beta^-(\gamma_1)$	$2{,}75 \pm 0{,}10$	$-95{,}1 \pm 1{,}1$	2493,9	232,15	8,291
	14	**Si**	**0**	**14**		γ_1	$1{,}80 \pm 0{,}10$	$-144{,}0 \pm 2{,}4$	2534,6	235,94	8,426
29	13	*Al*	3	16	6,7 m	β^-	2,5	$-126{,}1 \pm 5$	2614,3	243,36	8,392
	14	**Si**	**1**	**15**				$-153{,}1 \pm 1{,}4$	2633,2	245,12	8,452
	15	*P*	−1	14	$4{,}6 \pm 0{,}2$ s	β^+	$3{,}63 \pm 0{,}07$	$-103{,}1 \pm 1{,}6$	2575,1	239,70	8,266
30	**14**	**Si**	**2**	**16**				$-169{,}0 \pm 3{,}2$	2738,5	254,92	8,497
	15	*P*	0	15	$130{,}6 \pm 1{,}5$ s	β^+	$3{,}0 \pm 0{,}1$	$-126{,}0 \pm 3{,}4$	2687,4	250,16	8,339
31	14	*Si*	3	17	$157{,}3 \pm 1{,}3$ m	β^-(no γ)	1,8	$-155{,}1 \pm 4$	2814,1	261,96	8,450
	15	**P**	**1**	**16**				$-174{,}4 \pm 1{,}9$	2825,3	262,99	8,484
	16	*S*	−1	15	$3{,}18 \pm 0{,}04$ s	β^+	$3{,}85 \pm 0{,}07$	$-121{,}7 \pm 3{,}7$	2764,4	257,33	8,301
32	15	*P*	2	17	$14{,}30 \pm 0{,}03$ d	β^-(no γ)	$1{,}712 \pm 0{,}008$	$-172{,}5 \pm 0{,}7$	2912,8	271,14	8,473
	16	**S**	**0**	**16**				$-191{,}1 \pm 0{,}7$	2923,3	272,12	8,504

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|E|		|E|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
33	**16**	**S**	**1**	**17**				-198,6 ± 4	3020,2	281,14	8,519
	17	*Cl*	–1	16	2,4 ± 0,2 *s*	$\beta+$	4,13 ± 0,07	-143,2 ± 4	2956,7	275,23	8,340
34	15	*P*	4	19	12,40 ± 0,12 *s*	75 % β^-	5,1	-176	3095	288,2	8,476
						25 % $\beta^-(\gamma?)$	3,2				
	16	**S**	**2**	**18**		γ'	3,4	-230,1 ± 2,0	3141,2	292,40	8,600
	17	*Cl*	0	17	33,2 ± 0,5 *m*	80 % $\beta+$	5,1 ± 0,3	-163 ± 3,6	3066	285,5	8,397
						20 % $\beta+(\gamma')$	2,4				
35	16	*S*	3	19	87,1 *d*	β^-	0,1691	-210,3 ± 2,0	3210,8	298,88	8,539
	17	**Cl**	**1**	**18**				-211,6 ± 1,9	3204,0	298,25	8,521
	18	*A*	–1	17	1,88 ± 0,04 *s*	$\beta+$	4,38 ± 0,07	-153,5 ± 2,1	3137,7	292,08	8,345
36	**16**	**S**	**4**	**20**				-220	3310	308,1	8,559
	17	*Cl*	2	19	≈ 10^6 *a*	β^-	0,64	-213,5 ± 3,5	3295,3	306,75	8,521
						$\beta+, K$					
	18	**A**	**0**	**18**				-221,0 ± 4,0	3294,7	306,69	8,519
37	16	*S*	(5)	(21)	5,04 ± 0,02 *m*	10 % β^-	4,3	-177	3356	312,5	8,446
						90 % $\beta^-(\gamma_1)$	1,6				
	17	**Cl**	**3**	**20**		γ_1	2,8	-223,1 ± 1,0	3394,4	315,97	8,540
	18	*A*	1	19	34,1 ± 0,3 *d*	*K* (no γ)					
						β?	0,33				
38	17	*Cl*	4	21	38,5 ± 0,5 *m*	52 % β^-	4,99 ± 0,06	-200,3 ± 3,2	3461,0	322,18	8,478
						16 % $\beta^-(\gamma_1)$	2,82				
						32 % $\beta^-(\gamma_2\gamma_1)$	1,15				
	18	**A**	**2**	**20**		γ_1	2,15 ± 0,05	-253,9 ± 1,3	3506,5	326,41	8,583
						γ_2	1,65 ± 0,05				
	19	*K*	0	19	7,65 ± 0,1 *m*	$\beta+(\gamma_1)$	2,53	-191,8	3436,3	319,95	8,420
39	**19**	**K**	**1**	**20**				-250 — ε	3583,9	333,61	8,554
40	**18**	**A**	**4**	**22**		γ	1,55 ± 0,05	-245,1 ± 1,2	3676,6	342,24	8,556
	19	*K*	2	21	2,4 ± 1 . 10^8 *a*	30 % β^-	1,35 ± 0,05	-244 — ε	3667,3	341,38	8,535
						70 % $K(\gamma)$					
	20	**Ca**	**0**	**20**				-249 — ε	3672,9	341,98	8,539
41	18	*A*	5	23	109,4 ± 1 *m*	0,7 % β^-	2,55	-226,0	3746,9	348,79	8,507
						99,3 % $\beta^-(\gamma_1)$	1,18 ± 0,03				
	19	**K**	**3**	**22**		γ_1	1,37 ± 0,06	-252,5	3765,3	350,59	8,551
						γ'	1,1 ± 0,1				
	20	*Ca*	(1)	(21)	8,5 ± 0,8 *d*	$K(\gamma')$		-260 — ε	3764,3	350,49	8,549
	21	*Sc*	–1	20	0,87 ± 0,03 *s*	$\beta+$	4,94 ± 0,07				
42	19	*K*	4	23	12,4 ± 0,2 *h*	70 % β^-(no γ)	3,50 ± 0,12	-244 — ε	3846,6	358,16	8,528
						30 % $\beta^-(\gamma ?)$	2,04				
	20	**Ca**	**2**	**22**		γ	1,51	-283 — ε	3877,1	360,91	8,593
43	**20**	**Ca**	**3**	**23**		γ	1,65	-278 — ε	3961,2	368,83	8,577
	21	*Sc*	1	22	3,92 ± 0,02 *h*	80 % $\beta+$	1,13 ± 0,05	-255 — ε	3930,0	365,92	8,510
						20 % $K(\gamma)$					

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	|ℰ| 10^{-4} MU	|ℰ| MeV	|ℰ|/A MeV
1	2	3	4	5	6	7	8	9	10	11	12
44	20	**Ca**	4	24		γ	1,33				
	21	*Sc*	2	23	3,92 ± 0,03 *h*	25 % β^+	1,33				
						75 % $K(\gamma)$					
	21	*Sc**	2	23	2,44 ± 0,03 *d*	*I. T.* γ	0,268 ± 0,005				
45	20	*Ca*	5	25	152 *d*	β^-(no γ)	0,260 ± 0,005	-321 ± 6	4183,4	389,52	8,656
	21	**Sc**	3	24				-325 ± 6	4179,3	389,04	8,645
	22	*Ti*	1	23	3,08 ± 0,06 *h*	β^+	1,24	-302,4 ± 7	4148,6	386,18	8,582
46	20	**Ca**	6	26							
	21	*Sc*	4	25	85 ± 1 *d*	6 % $\beta^-(\gamma_1)$	1,49 ± 0,01	-318,5 ± 6,3	4262,3	396,86	8,627
						94 % $\beta^-(\gamma_2\gamma_1)$	0,36 ± 0,01				
	22	**Ti**	2	24		γ_1	0,88 ± 0,02	-345,1 ± 6,2	4280,7	398,48	8,663
						γ_2	1,12 ± 0,02				
47	20	*Ca*	(7)	(27)	5,8 *d*	$\beta^-(\gamma)$	1,1				
	21	*Sc*	(5)	(26)	3,40 *d*	γ	1,3	-346,9	4380,1	407,79	8,676
						β^-(no γ)	0,46				
	22	**Ti**	3	25				-352,6 ± 9,4	4377,7	407,50	8,670
48	20	**Ca**	8	28							
	21	*Sc*	6	27	1,81 ± 0,02 *d*	$\beta^-(\gamma_2\gamma_1)$	0,640 ± 0,007	-359,0	4481,7	417,24	8,694
	22	**Ti**	4	26		γ_1	1,33 ± 0,03	-366,4 ± 4,7	4480,9	417,12	8,690
						γ_2	0,98 ± 0,02				
	23	*V*	2	25	16,0 ± 0,2 *d*	58 % $\beta^+(\gamma_2\gamma_1)$	0,716 ± 0,015	-321,8	4428,2	412,31	8,590
						42 % K					
49	20	*Ca*	9	29	2,5 ± 0,1 *h*	$\beta^-(\gamma)$	2,3 ± 0,1				
	20	*Ca*(*)	(9)	(29)	30 *m*	β^-					
	21	*Sc*	7	28	57 ± 2 *m*	γ	0,8 ± 0,1	-332,4 ± 5,3	4544,6	423,05	8,634
						β^-(no γ)	1,8 ± 0,1				
	22	**Ti**	5	27				-351,8 ± 5,2	4555,8	424,08	8,655
	23	*V*	3	26	600 ± 50 *d*	γ	0,19				
						γ	1,55				
						K(no β^+ no γ)					
	23	*V**	3	26	33 ± 1 *m*	β^+	1,9				
	24	*Cr*	1	25	41,9 ± 0,3 *m*	$\beta^+(\gamma)$	1,45				
50	22	**Ti**	6	28				-376,8 ± 3,8	4670,2	434,74	8,695
	23	*V*	4	27	3,7 ± 0,2 *h*	β^+					
	24	**Cr**	2	26							
51	22	*Ti*	7	29	72 ± 2 *d*	$\beta^-(\gamma_3)$	0,36	-405,3	4788,2	445,78	8,741
	22	*Ti*(*)	7	29	2,9 ± 0,1 *m*	$\beta^-(\gamma)$					
	23	**V**	5	28		γ_3	1,02	-420,9 ± 6,9	4795,6	446,41	8,753
						γ_1	0,237 ± 0,001				
						γ_2	0,330 ± 0,001				
	24	*Cr*	3	27	26,4 ± 1,0 *d*	97 % K(no γ)		-420	4787	445,6	8,737
						≈0,3 % $K(\gamma_1)$					
						≈ 3 % $K(\gamma_2)$					
	25	*Mn*	1	26	46 ± 2 *m*	β^+	2,0				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	|𝓔| 10^{-4} MU	|𝓔| MeV	|𝓔|/A MeV
1	2	3	4	5	6	7	8	9	10	11	12
52	23	V	6	29	3,74 ± 0,01 m	$\beta^-(\gamma_1)$	2,05	-438,8 ± 7,5	4903,0	456,40	8,777
	24	Cr	4	28		γ_1	1,46 ± 0,03	-475,4	4931,5	459,17	8,830
						γ_2	0,734 ± 0,015				
						γ_3	0,940 ± 0,021				
	25	Mn	2	27	6,5 ± 1 d	$\beta^+(\gamma_2\gamma_3\gamma_1)$	0,582 ± 0,015	-424,5	4872,4	453,67	8,724
						65 % $K(\gamma)$					
	25	Mn*	2	27	21 ± 2 m	$\beta^+(\gamma_1)$	2,66 ± 0,05				
						0,5 % I.T. γ	0,392 ± 0,008				
	26	Fe	0	26	7,8 h	$\beta+$	0,55				
53	24	Cr	5	29				-440	4985	464,0	8,755
	26	Fe	1	27	8,9 ± 0,2 m	$\beta+$		-336 ± 22	4865	453,0	8,547
54	24	Cr	6	30		γ	0,835 ± 0,015	-397 ± 19	5032	468,4	8,674
	25	Mn	4	29	310 ± 20 d	$K(\gamma)$					
	26	Fe	2	28				-400 ± 20	5019	467,2	8,652
55	24	Cr	(7)	(31)	≈ 2 h	β^-					
	25	Mn	5	30		$\gamma'?$	0,070	-403 ± 17	5118,9	476,62	8,666
	26	Fe	3	29	≈ 4 a	γ	0,16				
						γ	0,21				
						γ	0,8				
						γ	1,2				
						$K(\gamma'?)$					
	27	Co	1	28	18,2 h	$\beta+(\gamma)$	1,50				
56	25	Mn	6	31	2,59 ± 0,02 h	60 % $\beta^-(\gamma_1)$	2,86 ± 0,05	-388 ± 17	5193,5	483,57	8,635
						25 % $\beta^-(\gamma_2\gamma_1)$	1,05 ± 0,03				
						15 % $\beta^-(\gamma_3\gamma_1)$	0,75 ± 0,1				
	26	Fe	4	30		γ_1	0,845 ± 0,015	-429 ± 17	5226,6	486,52	8,688
						γ_2	1,81 ± 0,04				
						γ_3	2,13 ± 0,05				
						γ_4	1,24 ± 0,04				
						γ	1,74				
						γ	2,01				
						γ	2,55				
						γ	3,25				
	27	Co	2	29	72 d	$\beta+(\gamma_1\gamma_1)$	1,50 ± 0,05	-378 ± 17	5167,7	481,16	8,592
						$K(\gamma)$					
57	26	Fe	5	31		γ	0,117 ± 0,001	-430	5317	495,1	8,686
						γ	0,130 ± 0,001				
						γ	0,202 ± 0,004				
						γ	0,215 ± 0,004				
	27	Co	3	30	270 d	$\beta+$	0,26				
						$K(\gamma)$					
	28	Ni	1	29	36,9 ± 1,5 h	$\beta+$	0,67 ± 0,1				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	$W-A$	$\lvert\mathcal{E}\rvert$		$\lvert\mathcal{E}\rvert/A$
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
58	**26**	**Fe**	**6**	**32**		γ_1	$0{,}805 \pm 0{,}012$				
	27	Co	4	31	72 d	10 % $\beta^+(\gamma_1)$	$0{,}470 \pm 0{,}015$				
						90 % $K(\gamma_1)$					
	28	**Ni**	**2**	**30**				$-402{,}9 \pm 4{,}1$	5363,1	499,23	8,607
	29	*Cu*	(0)	(29)	10 m	β^+					
59	26	*Fe*	7	33	44 d	50 % $\beta^-(\gamma_1)$	$0{,}257 \pm 0{,}008$				
						50 % $\beta^-(\gamma_2)$	$0{,}460 \pm 0{,}007$				
	27	**Co**	**5**	**32**		γ_1	$1{,}30 \pm 0{,}02$	$-475{,}2 \pm 4$	5532,9	515,17	8,732
						γ_2	$1{,}10 \pm 0{,}02$				
	28	*Ni*	(3)	(31)	15 a	β^+	0,05				
	29	*Cu*	(1)	(30)	$81 \pm 2\,s$	β^+					
60	27	Co	6	33	5,3 a	$\beta^-(\gamma_1\,\gamma_2)$	$0{,}308 \pm 0{,}008$	$-471{,}3 \pm 3{,}5$	5618,5	523,14	8,719
	27	Co*	6	33	10,7 m	90 % $I.T.$ γ	$0{,}056 \pm 0{,}003$				
						10 % $\beta^-(\gamma_3)$	$1{,}25 \pm 0{,}06$				
	28	**Ni**	**4**	**32**		γ_1	$1{,}30 \pm 0{,}03$	$-501{,}9 \pm 3{,}4$	5641,0	525,10	8,752
						γ_2	$1{,}10 \pm 0{,}02$				
						γ_3	$1{,}50 \pm 0{,}05$				
	29	*Cu*	2	31	$24{,}6 \pm 0{,}3\,m$	5 % β^+	3,3	-454	5584,8	520,0	8,667
						$\beta^+(\gamma_3)$	1,8				
61	27	Co	7	34	$1{,}75 \pm 0{,}05\,h$	β^- (no γ)	1,1	-447	5684	529,2	8,68
	28	**Ni**	**5**	**33**				-460 ± 15	5688,5	529,53	8,681
	29	*Cu*	3	32	$3{,}4 \pm 0{,}1\,h$	β^+(no γ)	$1{,}225 \pm 0{,}015$	-435 ± 15	5654,9	526,53	8,632
						K(no γ)					
62	**28**	**Ni**	**6**	**34**				$-504{,}1 \pm 3{,}9$	5822,1	541,96	8,741
	29	*Cu*	4	33	$10{,}1 \pm 0{,}1\,m$	β^+	2,6				
63	28	*Ni*	7	35	long						
	29	**Cu**	**5**	**34**		γ_1	$0{,}960 \pm 0{,}008$	$-512{,}2 \pm 6{,}3$	5911,5	550,28	8,735
						γ_2	$1{,}89 \pm 0{,}06$				
						γ	$2{,}60 \pm 0{,}08$				
	30	*Zn*	3	33	$38{,}3 \pm 0{,}5\,m$	90 % β^+(no γ)	$2{,}36 \pm 0{,}04$	$-476{,}5 \pm 6{,}3$	5867,7	546,20	8,670
						7 % $\beta^+(\gamma_1)$	1,40				
						3 % $\beta^+(\gamma_2)$	0,5				
						$K(\gamma)$					
64	**28**	**Ni**	**8**	**36**		γ	$1{,}20 \pm 0{,}05$	$-525{,}6 \pm 5{,}6$	6022,5	560,61	8,760
	29	*Cu*	6	35	$12{,}8 \pm 0{,}3\,h$	65 % β^- (no γ)	$0{,}578 \pm 0{,}003$	$-507{,}4 \pm 5{,}6$	5996,1	558,16	8,721
						32 % β^+(no γ)	$0{,}659 \pm 0{,}003$				
						3 % $K(\gamma)$					
	30	**Zn**	**4**	**34**				$-513{,}6 \pm 5{,}6$	5994,2	557,98	8,718
	31	Ga	(2)	(33)	$48 \pm 2\,m$	β^+					
65	28	*Ni*	9	37	$2{,}6 \pm 0{,}3\,h$	$\beta^-(\gamma)$	1,9				
	29	**Cu**	**7**	**36**		γ	0,280	-532 ± 30	6110	568,8	8,750
						γ	0,65				
						γ	0,93				
						γ'	$1{,}14 \pm 0{,}02$				

A	*Z*	Symbol	*n*	*N*	Half-life	Decay	Energy of radiation MeV	*W—A* 10^{-4} MU	$\lvert\mathcal{E}\rvert$ 10^{-4} MU	$\lvert\mathcal{E}\rvert$ MeV	$\lvert\mathcal{E}\rvert/A$ MeV
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
	30	*Zn*	5	35	250 ± 5 *d*	γ	0,054	-516	6086	566,7	8,718
						γ	0,117				
						53 % *K* (no γ)					
						45 % *K* (γ')					
						2,2 % β^+	0,47				
	31	Ga	3	34	15 *m*	*K* (γ)					
66	28	*Ni*	10	38	56 *h*	β^-					
	29	*Cu*	8	37	5,05 ± 0,05 *m*	$\beta^-(\gamma_1)$	2,58				
	30	**Zn**	**6**	**36**		γ_1	1,32 ± 0,04				
	31	*Ga*	4	35	9,2 ± 0,2 *h*	β^+	3,9 ± 0,2				
67	29	*Cu*	(9)	(38)	56 *h*	β^-					
	30	**Zn**	**7**	**37**		γ	0,297				
						γ	0,180				
						γ	0,0925				
	31	*Ga*	5	36	3,26 ± 0,03 *d*	*K* (γ)					
68	**30**	**Zn**	**8**	**38**							
	31	*Ga*	6	37	66 ± 3 *m*	β^+	1,85				
	32	*Ge*	4	36	195 *d*	*K*					
69	30	*Zn*	9	39	57 ± 2 *m*	β^- (no γ)	0,86 ± 0,04	-455	6382	594,1	8,610
	30	*Zn**	9	39	13,8 ± 0,4 *h*	*I.T.* γ	0,439				
	31	**Ga**	**7**	**38**				-463	6383	594,2	8,612
	32	*Ge*	(5)	(37)	1,66 *d*	β^+					
70	**30**	**Zn**	**10**	**40**							
	31	*Ga*	8	39	19,8 ± 0,4 *m*	$\beta^-(\gamma)$	1,68				
	32	**Ge**	**6**	**38**							
71	**31**	**Ga**	**9**	**40**		γ	0,6	-476	6575	612,0	8,620
	32	*Ge*	7	39	11 *d*	*K* (γ)					
	32	*Ge**	7	39	37 ± 1,5 *h*	β^+	1,2				
	33	*As*	5	38	52 *m*	β^+					
	34	*Se*	3	37	44 *m*	β^+					
72	30	*Zn*	12	42	49 *h*	5 % β^-	1,6				
						95 % $\beta^-(\gamma)$	0,3				
	31	*Ga*	10	41	14,25 ± 0,20 *h*	9,5 % $\beta^-(\gamma_1)$	3,15				
						8 % $\beta^-(\gamma_2\gamma_1)$	2,52				
						10,5 % $\beta^-(\gamma_3\gamma_2\gamma_1)$	1,48				
						32 % $\beta^-(\gamma_4\gamma_1)$	0,955				
						40 % $\beta^-(\gamma_5\gamma_1)$	0,64				
	32	**Ge**	**8**	**40**		γ_1	0,835				
						γ_2	0,631				
						γ_3	1,05				
						γ_4	2,20				
						γ_5	2,51				
	32	*Ge**	8	40	5 ± 0,5 . 10^{-7} *s*	*I.T.* γ'	0,691				
	33	*As*	6	39	26 *h*	$\beta^+(\gamma)$	2,78 ± 0,10				
	34	*Se*	4	38	9,5 *d*	*K*					

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|𝓔|		|𝓔|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
73	30	*Zn*	(13)	(43)	< 2 *m*	β^-					
	31	*Ga*	(11)	(42)	5 *h*	β^- (no γ)	1,4				
	32	**Ge**	**9**	**41**		γ	0,052 ± 0,003				
	33	*As*	7	40	90 ± 10 *d*	$K(\gamma)$					
	34	*Se*	5	39	7,0 *h*	β^+ (no γ)	1,29				
						K					
74	**32**	**Ge**	**10**	**42**							
	33	*As*	8	41	17,5 ± 0,1 *d*	β^+ (γ?)	0,9				
						β^- (γ?)	1,3				
						γ'	0,582				
	34	**Se**	**6**	**40**							
	35	*Br*	4	39	106 *m*	β^+ (no γ)	1,5				
75	32	*Ge*	11	43	89 ± 6 *m*	$\beta^-(\gamma)$	1,2				
	33	**As**	**9**	**42**		γ'	0,35				
						γ'	0,18				
	34	*Se*	7	41	127 ± 2 *d*	$K(\gamma')$					
76	**32**	**Ge**	**12**	**44**							
	33	*As*	10	43	26,75 ± 0,15 *h*	60 % β^- (no γ)	2,76				
						25 % $\beta^-(\gamma_1)$	2,16				
						15 % $\beta^-(\gamma_2\,\gamma_1)$	0,94				
	34	**Se**	**8**	**42**		γ_1	0,57 ± 0,02				
						γ_2	1,25 ± 0,03				
						γ	1,84				
						γ	2,15				
77	32	*Ge*	13	45	12 *h*	$\beta^-(\gamma)$	2,0				
	32	*Ge*(*)	(13)	(45)	59 ± 2 *s*	β^-	2,8				
	33	*As*	(11)	(44)	40 *h*	β^-	0,7				
	34	**Se**	**9**	**43**							
	34	*Se**	9	43	17,5 ± 0,3 *s*	*I.T.* γ	0,15				
	35	*Br*	7	42	48 *h*						
	36	*Kr*	5	41	65 *m*	β^+	1,7				
						$K(\gamma)$					
78	33	*As*	12	45	65 ± 3 *m*	$\beta^-(\gamma')$	1,4				
	34	**Se**	**10**	**44**		γ'	0,27				
						γ	0,046				
						γ	0,108				
	35	*Br*	8	43	6,4 ± 0,1 *m*	$\beta^+(\gamma)$	2,3				
	36	**Kr**	**6**	**42**				−550 ± 16	7234	673,4	8,633
79	34	*Se*	(11)	(45)	< 10 *m* or > 7 . 10^6 *a*	β^-					
	35	**Br**	**9**	**44**							
	36	*Kr*	7	43	1,30 *d*	$\beta^+(\gamma)$	0,4 ± 0,1				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|E|	|E|	|E|/A
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
80	**34**	**Se**	**12**	**46**							
	35	*Br*	10	45	$18{,}5 \pm 0{,}5\,m$	$\beta^-(\gamma)$	$2{,}00 \pm 0{,}10$				
						$3\%\,\beta^+$	$0{,}73 \pm 0{,}10$				
	35	*Br**	10	45	$4{,}54 \pm 0{,}10\,h$	*I.T.* γ'	0,037				
						γ'	0,049				
	36	**Kr**	**8**	**44**		γ	$< 0{,}5$				
81	33	*As*	(15)	(48)	$< 10\,m$	β^-					
	34	*Se*	(13)	(47)	$18\,m$	β^-(no γ)	1,5				
	34	*Se**	(13)	(47)	$57 \pm 1\,m$	*I.T.* γ	0,098				
	35	**Br**	**11**	**46**							
82	**34**	**Se**	**14**	**48**							
	35	*Br*	12	47	$33{,}9 \pm 0{,}3\,h$	$\beta^-(\gamma)$	0,465				
	36	**Kr**	**10**	**46**		γ	0,547	-610 ± 12	7652	712,3	8,687
						γ	0,787				
						γ	1,35				
	37	*Rb*	(8)	(45)	$20\,m$						
83	34	*Se*	15	49	$30\,m$	$\beta^-(\gamma)$	1,5				
	34	*Se**	15	49	$67 \pm 3\,s$	$\beta^-(\gamma')$	3,4				
	35	*Br*	13	48	$140 \pm 10\,m$	γ	0,17				
						γ	0,37				
						γ	1,1				
						β^-(no γ)	1,05				
	36	**Kr**	**11**	**47**							
	36	*Kr**	11	47	$113\,m$	*I.T.* γ	0,029				
						γ	0,046				
84	34	*Se*	16	50	$\approx 2\,m$	β^-					
	35	*Br*	14	49	$33\,m$	$\beta^-(\gamma)$	5,3				
	36	**Kr**	**12**	**48**				-618 ± 13	7839	729,7	8,687
	37	*Rb*	(10)	(47)	$6{,}5\,h$						
	37	*Rb*(*)	(10)	(47)	$40\,d$	β^+					
	38	**Sr**	**8**	**46**							
85	35	*Br*	15	50	$3{,}0 \pm 0{,}5\,m$	β^-					
	36	*Kr*	13	49	≈ 10 a	β^-(no γ)	0,74				
	36	*Kr**	13	49	$4{,}5\,h$	$\beta^-(\gamma)$	0,94				
	37	**Rb**	**11**	**48**		γ	0,17				
						γ	0,37				
						γ'	0,8				
	38	*Sr*	9	47	$65\,d$	$K(\gamma')$					
	38	*Sr**	9	47	$70\,m$	*I.T.* γ	0,170				
86	**36**	**Kr**	**14**	**50**				-616 ± 13	8016	746,2	8,67'
	37	*Rb*	12	49	$19{,}5\,d$	$\beta^-(\gamma)$	$1{,}60 \pm 0{,}03$				
	38	**Sr**	**10**	**48**							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|δ|		|δ|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
87	35	*Br*	17	52	$50 \pm 10\,s$	β^-					
	36	*Kr*	15	51	$74\,m$	β^-	≈ 4				
	37	*Rb*	13	50	$5{,}8 \pm 1{,}0 \cdot 10^{10}\,a$	$\beta^-(\gamma)$	0,132				
						$\beta^-(\gamma)$	0,25				
	38	**Sr**	**11**	**49**		γ	0,034				
						γ	0,053				
						γ	0,082				
						γ	0,102				
						γ	0,129				
	38	*Sr**	11	49	$2{,}75 \pm 0{,}1\,h$	*I.T.* γ	0,386				
	39	*Y*	9	48	$80 \pm 3\,h$	K(no γ)					
	39	*Y**	(9)	(48)	$14 \pm 2\,h$	*I.T.* γ	0,5				
88	36	*Kr*	16	52	$2{,}92 \pm 0{,}16\,h$	β^-	2,5				
	37	*Rb*	14	51	$17{,}5 \pm 0{,}5\,m$	β^-	$5{,}06 \pm 0{,}15$				
	38	**Sr**	**12**	**50**		γ	2,8				
						γ	$1{,}87 \pm 0{,}05$				
						γ	0,908				
	39	*Y*	(10)	(49)	$87\,d$	$K(\gamma)$					
	39	*Y(*)*	10	49	$2{,}0 \pm 0{,}2\,h$	β^+	1,2				
89	36	*Kr*	17	53	$2{,}6\,m$	β^-					
	37	*Rb*	15	52	$15{,}4 \pm 0{,}2\,m$	β^-	$4{,}5 \pm 0{,}3$				
	38	*Sr*	13	51	$54\,d$	β^-(no γ)	1,51				
	39	**Y**	**11**	**50**							
	40	*Zr*	9	49	$78 \pm 1\,h$	β^+(no γ)	1,0				
	40	*Zr**	9	49	$4{,}5\,m$	*I.T.* or $K(\gamma)$					
90	36	*Kr*	18	54	$33\,s$	β^-					
	37	*Rb*	16	53	short	β^-					
	38	*Sr*	14	52	$25\,a$	β^-(no γ)	0,6				
	39	*Y*	12	51	$62\,h$	β^-(no γ)	2,45				
	40	**Zr**	**10**	**50**							
	41	*Cb*	(8)	(49)	$21\,h$	β^+	1				
91	36	*Kr*	19	55	$\approx 8\,s$	β^-					
	37	*Rb*	17	54	short	β^-					
	38	*Sr*	15	53	$9{,}7\,h$	60 % β^-	3,2				
						40 % $\beta^-(\gamma)$	1,3				
	39	*Y*	13	52	$57 \pm 3\,d$	β^-(no γ)	1,53				
	39	*Y**	13	52	$51\,m$	γ	$\approx 1{,}3$				
						I.T. γ'	0,61				
	40	**Zr**	**11**	**51**		γ	0,15				
						γ	0,94				
	41	*Cb*	(9)	(50)	$55\,d$	$K(\gamma)$					
92	36	*Kr*	20	56	$3\,s$	β^-					
	37	*Rb*	18	55	short	β^-					
	38	*Sr*	16	54	$2{,}7\,h$	β^-					
	39	*Y*	14	53	$3{,}5\,h$	$\beta^-(\gamma)$	3,5				
	40	**Zr**	**12**	**52**		γ	0,6				
	41	*Cb*	10	51	$11 \pm 1\,d$	β^-	1,38				
						$\beta^-(\gamma)$	0,59				
	42	**Mo**	**8**	**50**		γ	$\approx 1{,}0$				

A	Z	Symbol	n	N	Half-life	Decay		Energy of radiation	W—A	|δ|		|δ|/A
								MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7		8	9	10	11	12
93	36	*Kr*	21	57	2,0 *s*	β^-						
	37	*Rb*	19	56	short	β^-						
	38	*Sr*	17	55	7 *m*	β^-						
	39	*Y*	15	54	11,5 *h*	$\beta^-(\gamma)$		3,1				
		(*Zr*)			?		γ	0,7				
	41	**Cb**	**11**	**52**			γ'	1,6				
	41	*Cb**	11	52	55 ± 5 *d*	*I.T.*	γ	0,94				
							γ	0,15				
	42	*Mo*	9	51	6,70 ± 0,05 *h*	$\beta^+(\gamma')$		0,7				
						$\beta^+(\gamma')$		0,3				
94	36	*Kr*	(22)	(58)	1,4 *s*	β^-						
	37	*Rb*	(20)	(57)	short	β^-						
	38	*Sr*	(18)	(56)	≈ 2 *m*	β^-						
	39	*Y*	(16)	(55)	20 *m*	$\beta^-(\gamma)$						
	40	**Zr**	**14**	**54**								
	41	*Cb*	12	53	> 100 *a*							
	41	*Cb**	12	53	6,6 *m*	*I.T.*	γ	≈ 0,05				
						$\beta^-(\gamma')$		1,4				
	42	**Mo**	**10**	**52**			γ'	0,380 ± 0,004				
							γ	0,873 ± 0,004				
							γ	1,48 ± 0,01				
							γ	1,85 ± 0,01				
							γ	2,74 ± 0,01				
	43	*Tc*	(8)	(51)	< 50 *m*	30 % β^+		2,45 ± 0,03				
						70 % $K(\gamma\gamma')$						
	43	*Tc**	(8)	(51)	50 ± 2 *m*	*I.T.*	γ	0,0334				
95	39	*Y*	17	56	< 3 *h*	β^-						
	40	*Zr*	15	55	65 *d*	2 % β^-		1,0				
						98 % $\beta^-(\gamma)$		0,394				
	41	*Cb*	13	54	36,5 *d*		γ	0,73				
						$\beta^-(\gamma')$		0,154				
	41	*Cb**	13	54	80 *h*	*I.T.*	γ	0,24				
	42	**Mo**	**11**	**53**			γ'	0,776	-618	8773	816,9	8,599
							γ_1	0,201 ± 0,002				
							γ_2	0,810 ± 0,005				
							γ_3	0,570 ± 0,002				
							γ_4	1,017 ± 0,010				
	43	*Tc*	9	52	20,0 ± 0,5 *h*		γ	0,95				
						$K(\gamma')$						
	43	*Tc*(*)	(9)	(52)	62 *d*	0,8% β^+ (no γ)		≈ 0,4				
						29% K(no γ)						
						29% $K(\gamma_2\gamma_1)$						
						38% $K(\gamma_3\gamma_1)$						
						3% $K(\gamma_4)$						
	44	*Ru*	7	51	1,65 ± 0,05 *h*	β^+		1,1				
						$K(\gamma)$						

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	$W-A$ 10^{-4} MU	$\lvert\mathcal{E}\rvert$ 10^{-4} MU	$\lvert\mathcal{E}\rvert$ MeV	$\lvert\mathcal{E}\rvert/A$ MeV
1	2	3	4	5	6	7	8	9	10	11	12
96	**40**	**Zr**	**16**	**56**							
	42	**Mo**	**12**	**54**		γ	0,92				
	43	*Tc*	10	53	$4{,}2 \pm 0{,}1\,d$	$K(\gamma)$					
	43	*Tc*(*)	(10)	(53)	$2{,}7 \pm 0{,}4\,h$	$\beta+$					
	44	**Ru**	8	**52**							
97	36	*Kr*	25	61	short	β^-					
	37	*Rb*	23	60	short	β^-					
	38	*Sr*	21	59	short	β^-					
	39	*Y*	19	58	short	β^-					
	40	*Zr*	17	57	$17{,}0 \pm 0{,}2\,h$	$\beta^-(\gamma)$	2,1				
	41	*Cb*	15	56	$75 \pm 3\,m$	γ	$\approx 0{,}8$				
						$\beta^-(\gamma)$	1,4				
	42	**Mo**	**13**	**55**		γ	0,78	-631	8965	834,8	8,606
						γ'	0,097				
	43	*Tc*	11	54	$93 \pm 5\,d$	γ	0,23				
						$K(\gamma')$					
	44	*Ru*	9	53	$2{,}8 \pm 0{,}3\,d$	$K(\gamma)$					
98	**42**	**Mo**	**14**	**56**							
	43	*Tc*	(12)	(55)	$2{,}8 \pm 0{,}1\,d$	$\beta^-(\gamma)$	$1{,}3 \pm 0{,}2$				
	44	**Ru**	**10**	**54**		γ	$0{,}9 \pm 0{,}1$				
99	42	*Mo*	15	57	$67 \pm 2\,d$	90 % β^-(no γ)	1,03				
						10 % $\beta^-(\gamma_2\gamma_1)$	0,24				
	43	*Tc*	13	56	$9{,}4 \cdot 10^5\,a$	β^-(no γ)	0,32				
	43	*Tc**	13	56	$6{,}6 \pm 0{,}4\,h$	γ_2	0,71				
						I.T. γ_1	0,136				
	44	**Ru**	**11**	**55**				-643 ± 31	9140	850,8	8,594
100	**42**	**Mo**	**16**	**58**							
	44	**Ru**	**12**	**56**		γ	0,6				
						γ	1,2				
	45	*Rh*	(10)	(55)	$19{,}4\,h$	γ'	1,8				
						γ'	0,090				
						5 % $\beta+$	3,0				
						95 % $K(\gamma)$					
	46	*Pd*	(8)	(54)	$4{,}0\,d$	$K(\gamma')$					
101	42	*Mo*	17	59	$14{,}6 \pm 0{,}3\,m$	β^-	2,2				
						$\beta^-(\gamma)$	1,0				
	43	*Tc*	15	58	$14{,}0 \pm 0{,}3\,m$	γ	0,3				
						γ	0,9				
						$\beta^-(\gamma')$	1,2				
	44	**Ru**	**13**	**57**		γ'	0,30	-655 ± 31	9332	868,7	8,601
	45	*Rh*	(11)	(56)	$4{,}3\,d$	$K(\gamma')$					
	46	*Pd*	(9)	(55)	$9\,h$	10 % $\beta+$	$2{,}3 \pm 0{,}2$				
						90 % K(no γ)					

A	*Z*	Symbol	*n*	*N*	Half-life	Decay	Energy of radiation	*W—A*	|δ|		|δ|/*A*
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
102	44	**Ru**	14	58							
	45	*Rh*	12	57	200 ± 10 *d*	$\beta^+(\gamma)$	1,13 ± 0,08				
						β^-	1,04 ± 0,08				
	46	**Pd**	10	56							
103	44	*Ru*	15	59	42 *d*	5 % β^-	0,80				
						95 % $\beta^-(\gamma)$	0,2				
	45	**Rh**	13	58		γ	0,55	-550 ± 38	9397	874,7	8,493
	45	*Rh**	13	58	53 ± 3 *m*	*I.T.* γ'	0,020 ± 0,001				
	46	*Pd*	11	57	17,0 *d*	*K*					
104	44	**Ru**	16	60							
	45	*Rh*	14	59	41,8 ± 0,7 *s*	β^-	2,58 ± 0,05				
	45	*Rh**	14	59	4,34 ±0,06 *m*	*I.T.* γ	0,069				
	46	**Pd**	12	58							
105	42	*Mo*	21	63	short	β^-					
	43	*Tc*	19	62	short	β^-					
	44	*Ru*	17	61	4,4 ± 0,1 *h*	$\beta^-(\gamma)$	1,35				
	45	*Rh*	15	60	36,5 *h*	γ	0,76				
						$\beta^-(\gamma)$	0,60				
	46	**Pd**	13	59		γ	0,33				
106	44	*Ru*	18	62	1,0 *a*	β^- (no γ)	≈ 0,03				
	45	*Rh*	16	61	30 *s*	82 % β^- (no γ)	3,55 ± 0,10				
						18 % $\beta^-(\gamma')$	2,30 ± 0,10				
	46	**Pd**	14	60		γ'	0,51 ± 0,02	-570 ± 32	9678	900,8	8,499
						γ'	1,25 ± 0,05				
						γ' and γ	0,72 ± 0,02				
						γ	1,06 ± 0,03				
						γ	1,63 ± 0,04				
	47	*Ag*	12	59	8,2 ± 0,2 *d*	$K(\gamma)$					
	47	*Ag*(*)	12	59	25,0 ± 0,5 *m*	β^+ (no γ)	2,04 ± 0,05				
	48	**Cd**	10	58							
107	46	*Pd*	15	61	very short or > 3 . 10^8 *a*	β^-					
	47	**Ag**	13	60				-541 ± 20	9730	905,7	8,465
	47	*Ag**	13	60	44,3 ± 0,2 *s*	γ_2	0,846				
						I.T. γ_1	0,0935				
	48	*Cd*	11	59	6,7 *h*	0,31 % $\beta^+(\gamma_1)$	0,320 ± 0,010				
						0,42 % $K(\gamma_1)$					
						99,27% $K(\gamma_2\gamma_1)$					
108	46	**Pd**	16	62							
	47	*Ag*	14	61	2,44 ±0,06 *m*	β^-	2,06				
	48	**Cd**	12	60		γ	0,65				
	49	*In*	(10)	(59)	≈ 5 *h*	$K(\gamma)$					

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	$\lvert\delta\rvert$ 10^{-4}MU	$\lvert\delta\rvert$ MeV	$\lvert\delta\rvert/A$ MeV
1	2	3	4	5	6	7	8	9	10	11	12
109	45	*Rh*	19	64	$<1\,h$	β^-					
	46	*Pd*	17	63	13,2 h	β^-(no γ)	1,03				
	47	**Ag**	**15**	**62**				-552 ± 20	9919	923,3	8,471
	47	*Ag**	15	62	40,5 ± 0,7 s	*I.T.* γ	0,0884				
	48	*Cd*	13	61	158 d	γ	0,5				
						$K(\gamma)$					
	49	*In*	11	60	6,5 h	$\approx 2\,\%\,\beta^+$	2				
						$K(\gamma)$					
110	**46**	**Pd**	**18**	**64**		γ	1,51 ± 0,04	-592 ± 33	10057	936,1	8,511
						γ	0,925 ± 0,03				
						γ	0,650 ± 0,02				
	47	*Ag*	(16)	(63)	225 ± 10 d	$K(\gamma)$					
	47	*Ag*(*)	16	63	24,5 ± 0,3 s	β^-	2,7				
	48	**Cd**	**14**	**62**							
	49	*In*	12	61	66 ± 5 m	β^+	1,6 ± 0,3				
111	46	*Pd*	19	65	26 m	β^-	3,5				
	47	*Ag*	17	64	7,6 d	β^-(no γ)	0,80				
	48	**Cd**	**15**	**63**		γ'	0,173				
						γ'	0,247				
	48	*Cd**	15	63	48,7 ± 0,3 m	*I.T.* γ	0,145				
						γ	0,230				
	49	*In*	13	62	2,7 d	$K(\gamma')$					
112	46	*Pd*	20	66	21 h	β^-(no γ)	0,2				
	47	*Ag*	18	65	3,2 h	β^-	3,6				
						$\beta^-(\gamma)$	2,2				
	48	**Cd**	**16**	**64**		γ	0,86				
	49	*In*	14	63	9 m	β^-	1				
						β^+	1,7				
						$K(\gamma')$					
	49	*In**	14	63	23 m	*I.T.* γ	0,16 ± 0,01				
	50	**Sn**	**12**	**62**							
113	**48**	**Cd**	**17**	**65**							
	48	*Cd**	17	65	2,3 m						
	49	**In**	**15**	**64**		γ'	0,085 ± 0,01				
	49	*In**	15	64	104 ± 2 m	*I.T.* γ	0,39 ± 0,02				
	50	*Sn*	13	63	105 ± 15 d	$K(\gamma')$					
114	**48**	**Cd**	**18**	**66**							
	49	*In*	16	65	72 s	β^-	1,98 ± 0,03				
	49	*In**	16	65	48,5 ± 2 d	*I.T.* γ	0,192 ± 0,001				
	50	**Sn**	**14**	**64**							
115	48	*Cd*	19	67	2,33 d	40 % β^-	1,20				
						60 % $\beta^-(\gamma_1)$	0,56				
	48	*Cd**	19	67	43 ± 3 d	$\beta^-(\gamma_1?)$	1,7				
	49	**In**	**17**	**66**		γ_1	0,65 ± 0,02				
	49	*In**	17	66	4,53 ± 0,03 h	*I.T.* γ	0,338 ± 0,001				
	50	**Sn**	**15**	**65**							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|δ|		|δ|/A
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
116	**48**	**Cd**	**20**	**68**							
	49	*In*	18	67	13 *s*	β^-(no γ)	2,8				
	49	*In*(*)	18	67	54,05 ± 0,16 *m*	β^-(γ)	0,85 ± 0,01				
	50	**Sn**	**16**	**66**		γ	2,32 ± 0,07	-609 ± 46	10578	984,7	8,489
						γ	1,85 ± 0,05				
						γ	1,35 ± 0,04				
						γ	1,08 ± 0,04				
						γ	0,57 ± 0,03				
						γ	0,36 ± 0,03				
						γ	0,17 ± 0,03				
117	48	*Cd*	21	69	2,72 ± 0,2 *h*	β^-	≈ 1,5				
	49	*In*	19	68	1,95 ± 0,05 *h*	β^-(no γ)	1,73				
	50	**Sn**	**17**	**67**							
	51	*Sb*	(15)	(66)	2,8 ± 0,3 *h*	*K*					
118	**50**	**Sn**	**18**	**68**		γ	1,5	-620 ± 47	10768	1002,4	8,495
	51	*Sb*	(16)	(67)	5,1 ± 0,3 *h*	*K*(γ)					
	51	*Sb*(*)	16	67	3,5 *m*	β^+	3,1 ± 0,2				
	52	*Te*	14	66	6,0 *d*	*K*(no γ)					
119	**50**	**Sn**	**19**	**69**				-625 ± 48	10862	1011,1	8,497
	51	*Sb*	17	68	39 ± 1 *h*	γ	0,2				
						γ	0,5				
						γ	1,4				
						K(no γ)					
	52	*Te*	15	67	4,5 *d*	*K*(γ)					
120	**50**	**Sn**	**20**	**70**		γ	1,1				
	51	*Sb*	18	69	6,0 *d*	*K*(γ)					
	51	*Sb*(*)	18	69	15 ± 0,8 *m*	β^+	1,53				
	52	**Te**	**16**	**68**							
121	50	*Sn*	21	71	28 *h*	β^-(no γ)	0,4				
	51	**Sb**	**19**	**70**		γ	0,61				
	52	*Te*	17	69	17 ± 1 *d*	*K*(γ)					
	52	*Te**	17	69	5 ± 2.10^{-8} *s*	*I.T.* γ	0,233				
	52	*Te***	17	69	143 ± 5 *d*	*I.T.* γ	0,050				
122	**50**	**Sn**	**22**	**72**				-543 ± 50	11049	1028,5	8,430
	51	*Sb*	20	71	63 *h*	β^-	1,76 ± 0,10				
						β^-(γ)	1,19 ± 0,05				
	51	*Sb**	20	71	3,5 *m*	*I.T.* γ'	0,146				
	52	**Te**	**18**	**70**		γ	0,57				
123	**51**	**Sb**	**21**	**72**							
	52	**Te**	**19**	**71**							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	|𝓔| 10^{-4}MU	|𝓔| MeV	|𝓔|/A MeV
1	2	3	4	5	6	7	8	9	10	11	12
124	**50**	**Sn**	**24**	**74**				-552 ± 51	11237	1046,0	8,453
	51	*Sb*	22	73	60 *d*	21% $\beta^-(\gamma_1)$	2,37				
						8% $\beta^-(\gamma_2\gamma_1)$	1,62				
						9% $\beta^-(\gamma_3\gamma_2\gamma_1)$	1,00				
						44% $\beta^-(\gamma_4\gamma_1)$	0,65				
						18% $\beta^-(\gamma)$	0,42				
	51	*Sb**	22	73	1,3 *m*	$\beta^-(\gamma')$	3,2 ± 0,2				
	51	*Sb***	22	73	21 *m*	*I.T.* γ''	0,020				
	52	**Te**	**20**	**72**		γ_1	0,603				
						γ_2	0,714				
						γ_3	0,650				
						γ_4	1,708				
						γ	2,062				
	52	*Te**	20	72	12 ± 4 . 10^{-4} *s*	*I.T.* γ'''	0,069 ?				
	53	*I*	18	71	4,0 ± 0,3 *d*	$\beta+$					
	54	**Xe**	**16**	**70**							
125	50	*Sn*	(25)	(75)	11,8 ± 0,5 *m*	β^-					
	51	*Sb*	(23)	(74)	≈ 2,7 *a*	35 % β^-	0,7				
						65 % $\beta^-(\gamma)$	0,3				
	52	**Te**	**21**	**73**		γ	0,56				
	53	*I*	(19)	(72)	56 *d*	*K*(no β no γ)					
126	**52**	**Te**	**22**	**74**							
	53	*I*	20	73	13,3 ± 0,3 *d*	$\beta^-(\gamma)$	1,20 ± 0,03				
						K					
	54	**Xe**	**18**	**72**		γ	0,5				
127	51	*Sb*	25	76	93 *h*	$\beta^-(\gamma)$	1,15				
	52	*Te*	23	75	9,3 ± 0,5 *h*	γ	0,72				
						β^-(no γ)	0,70				
	52	*Te**	23	75	90 ± 2 *d*	*I.T.* γ	0,086				
	53	**I**	**21**	**74**		γ	0,9				
	54	*Xe*	(19)	(73)	34 *d*	*K*?(γ)					
	54	*Xe**	(19)	(73)	75 *s*	*I.T.* γ	0,175				
						γ	0,125				
128	**52**	**Te**	**24**	**76**							
	53	*I*	22	75	24,98 ± 0,02 *m*	93 % β^-	2,02				
						7 % $\beta^-(\gamma_1)$					
	54	**Xe**	**20**	**74**		γ_1	0,428				
129	51	*Sb*	27	78	4,2 *h*	β^-					
	52	*Te*	25	77	72 ± 3 *m*	$\beta^-(\gamma)$	1,75 ± 0,1				
	52	*Te**	25	77	32 ± 2 *d*	*I.T.* γ'	0,102				
	53	*I*	23	76	≳ 10^8 *a*	γ	0,8				
						γ	0,3				
						β^-					
	54	**Xe**	**21**	**75**				-546 ± 13	11645	1084,0	8,403

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	$W-A$	$\|\delta\|$		$\|\delta\|/A$
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
130	**52**	**Te**	**26**	**78**							
	53	*I*	24	77	$12{,}5 \pm 0{,}5\,h$	$60\%\beta^-(\gamma_1\gamma_2\gamma_3)$	$1{,}03 \pm 0{,}02$				
						$40\%\beta^-$ $(\gamma_4\gamma_1\gamma_2\gamma_3)$	$0{,}61 \pm 0{,}02$				
	54	**Xe**	**22**	**76**		γ_1	$0{,}537 \pm 0{,}005$				
						γ_2	$0{,}667 \pm 0{,}008$				
						γ_3	$0{,}744 \pm 0{,}010$				
						γ_4	$0{,}417 \pm 0{,}005$				
	55	*Cs*	(20)	(75)	$30\,m$						
	56	**Ba**	**18**	**74**							
131	52	*Te*	27	79	$25 \pm 5\,m$	β^-					
	52	*Te**	27	79	$29 \pm 5\,h$	*I.T.* γ	$0{,}177$				
	53	*I*	25	78	$8{,}0 \pm 0{,}2\,d$	$\beta^-(\gamma_1\gamma_2)$	$0{,}595 \pm 0{,}010$				
	54	**Xe**	**23**	**77**		γ_1	$0{,}367 \pm 0{,}007$				
						γ_2	$0{,}080 \pm 0{,}001$				
						γ'	$0{,}145 \pm 0{,}010$				
	55	*Cs*	21	76	$10{,}2\,d$	γ	$0{,}22 \pm 0{,}01$				
						γ	$0{,}500 \pm 0{,}015$				
						γ	$1{,}7 \pm 0{,}1$				
						$K(\gamma')$					
	56	*Ba**	19	75	$11{,}7 \pm 0{,}3\,d$	$K(\gamma)$					
132	51	*Sb*	30	81	$\approx 5\,m$	β^-					
	52	*Te*	28	80	$70\,h$	$\beta^-(\gamma)$	$0{,}28$				
	53	*I*	26	79	$2{,}3\,h$	γ	$0{,}22$				
						$50\,\%\,\beta^-$	$2{,}1$				
						$50\,\%\,\beta^-(\gamma\gamma')$	$1{,}0$				
	54	**Xe**	**24**	**78**		γ	$1{,}4$	-550	11918	1109,4	8,405
						γ'	$0{,}68$				
	55	*Cs*	(22)	(77)	$7{,}1\,d$	$K(\gamma')$					
	56	**Ba**	**20**	**76**							
133	51	*Sb*	31	82	$< 10\,m$	β^-					
	52	*Te*	29	81	$60\,m$	β^-					
	53	*I*	27	80	$22\,h$	$\beta^-(\gamma)$	$1{,}2$				
	54	*Xe*	25	79	$5{,}4\,d$	γ	$0{,}55$				
						$\beta^-(\gamma)$	$0{,}33$				
	55	**Cs**	**23**	**78**		γ	$0{,}084$				
						γ'	$0{,}320 \pm 0{,}010$				
	56	*Ba*	21	77	$> 20\,a$	$K(\gamma\gamma')$					
	56	*Ba**	21	77	$38{,}8\,h$	*I.T.* γ	$0{,}2764$				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	$\lvert\delta\rvert$ 10^{-4} MU	$\lvert\delta\rvert$ MeV	$\lvert\delta\rvert/A$ MeV
1	2	3	4	5	6	7	8	9	10	11	12
134	51	*Sb*	(32)	(83)	< 10 *m*	β^-					
	52	*Te*	(30)	(82)	43 *m*	β^-					
	53	*I*	(28)	(81)	54 *m*	$\beta^-(\gamma)$					
	54	**Xe**	**26**	**80**		γ	> 1				
	55	Cs	24	79	1,7 ± 0,1 *a*	75% $\beta^-(\gamma_1\gamma_2$ or $\gamma_3)$	0,645 ± 0,02				
						25% $\beta^-(\gamma_4\gamma_1\gamma_2)$					
	55	Cs*	24	79	3,0 ± 0,3 *h*	*I.T.* γ	0,16				
						β^-	0,8				
	56	**Ba**	**22**	**78**		γ_1	0,602 ± 0,012				
						γ_2	0,794 ± 0,015				
						γ_3	1,35 ± 0,03				
						γ_4	0,568 ± 0,015				
135	52	*Te*	31	83	< 2 *m*	β^-					
	53	*I*	29	82	6,6 ± 0,3 *h*	β^-	1,35				
	54	*Xe*	27	81	9,4 *h*	γ	1,5				
						$\beta^-(\gamma)$	0,94				
	54	*Xe**	27	81	15,6 *m*	*I.T.* γ	0,54				
	55	Cs	25	80	> 2,5 . 10^4 *a*	γ	0,26				
						β^-					
	56	**Ba**	**23**	**79**							
136	**54**	**Xe**	**28**	**82**							
	56	**Ba**	**24**	**80**							
	58	**Ce**	**20**	**78**							
137	53	*I*	(31)	(84)	30 ± 6 *s*	β^-					
	54	*Xe*	(29)	(83)	3,4 *m*	β^-	≈ 4				
	55	*Cs*	27	82	33 *a*	$\beta^-(\gamma_1)$	0,550 ± 0,005				
	56	**Ba**	**25**	**81**		γ'	0,88 ± 0,1				
	56	*Ba**	25	81	158 ± 5 *s*	*I.T.* γ_1	0,663 ± 0,006				
	57	*La*	(23)	(80)	17,5 ± 0,5 *h*	$K(\gamma')$					
138	54	*Xe*	(30)	(84)	17 ± 1 *m*	β^-					
	55	*Cs*	(28)	(83)	32 ± 0,5 *m*	$\beta^-(\gamma)$	2,6				
	56	**Ba**	**26**	**82**		γ	1,2				
	57	**La**	**24**	**81**							
	58	**Ce**	**22**	**80**							
139	54	*Xe*	31	85	41 *s*	β^-					
	55	*Cs*	29	84	7 *m*	β^-					
	56	*Ba*	27	83	85,6 *m*	$\beta^-(\gamma)$	2,2				
	57	**La**	**25**	**82**		γ	0,6	-478 ± 35	12448	1158,7	8,336
						γ'	0,21				
	58	Ce	(23)	(81)	140 ± 1 *d*	$K(\gamma')$					
140	54	*Xe*	32	86	9,8 *s*	β^-					
	55	*Cs*	30	85	short	β^-					
	56	*Ba*	28	84	12,8 *d*	75 % β^-	1,05 ± 0,05				
						25 % $\beta^-(\gamma)$	≈ 0,4				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|𝓔|		|𝓔|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
	57	*La*	26	83	$40{,}0 \pm 0{,}3\,h$	γ	0,535				
						$10\,\%\,\beta^-$	$2{,}12 \pm 0{,}08$				
						$70\,\%\,\beta^-(\gamma_1)$	$1{,}40 \pm 0{,}03$				
						$20\,\%\,\beta^-(\gamma)$	$0{,}90 \pm 0{,}03$				
	58	**Ce**	**24**	**82**		γ	0,333				
						γ	0,505				
						γ	0,832				
						γ_1	1,61				
						γ	2,52				
	59	*Pr*	22	81	$3{,}4 \pm 0{,}1\,m$	β^+	$2{,}40 \pm 0{,}15$				
141	54	*Xe*	33	87	$1{,}7\,s$	β^-					
	55	Cs	31	86	short	β^-					
	56	*Ba*	(29)	(85)	$18\,m$	$\beta^-(\gamma)$					
	57	*La*	(27)	(84)	$3{,}7\,h$	$\beta^-(\gamma)$	2,8				
	58	Ce	25	83	$30{,}6 \pm 0{,}7\,d$	$\beta^-(\gamma\gamma')$	0,65				
	59	**Pr**	**23**	**82**		γ	0,1371				
						γ	0,1450				
	59	*Pr**	23	82	$7 \pm 2 \cdot 10^{-5}\,s$	*I.T.* γ'	0,2				
	60	*Nd*	(21)	(81)	$1{,}6 \pm 0{,}3\,h$	β^+	0,78				
142	**58**	**Ce**	**26**	**84**		γ	1,9				
	59	*Pr*	24	83	$19{,}3 \pm 0{,}1\,h$	$\beta^-(\gamma')$	$2{,}14 \pm 0{,}02$				
						$K?(\gamma)$					
	60	**Nd**	**22**	**82**		γ'	1,0				
143	54	*Xe*	35	89	$1\,s$	β^-					
	55	Cs	33	88	short	β^-					
	56	*Ba*	31	87	$< 0{,}5\,m$	β^-					
	57	*La*	29	86	$19\,m$	β^-					
	58	Ce	27	85	$33\,h$	$\beta^-(\gamma)$	1,35				
	59	*Pr*	25	84	$13{,}8\,d$	γ	0,5				
						$\beta^-(\text{no}\,\gamma)$	0,95				
	60	**Nd**	**23**	**83**							
144	54	*Xe*	36	90	short	β^-					
	55	Cs	34	89	short	β^-					
	56	*Ba*	32	88	short	β^-					
	57	*La*	30	87	short	β^-					
	58	*Ce*	28	86	$275\,d$	$\beta^-(\text{no}\,\gamma)$	0,348				
	59	*Pr*	26	85	$17{,}5\,m$	$\beta^-(\gamma)$	3,07				
	60	**Nd**	**24**	**84**		γ	0,135				
						γ	1,25				
	62	**Sm**	**20**	**82**							
145	**60**	**Nd**	**25**	**85**							
	62	*Sm*	21	83	$\gtrsim 72\,d$						
146	**60**	**Nd**	**26**	**86**				-387 ± 30	12958	1206,2	8,262

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|𝓔|		|𝓔|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
147	60	*Nd*	27	87	11,0 *d*	60 % β^-	0,90				
						40 % $\beta^-(\gamma)$	≈ 0,4				
	61	*Pm*	25	86	3,7 *a*	γ	0,58				
						β^- (no γ)	0,20				
	62	**Sm**	**23**	**85**							
148	**60**	**Nd**	**28**	**88**				-371 ± 20	13122	1221,5	8,253
	61	*Pm*	26	87	5,3 *d*	$\beta^-(\gamma)$	2,5				
	62	**Sm**	**24**	**86**		γ	0,8				
149	61	*Pm*	27	88	47,5 ± 1,5 *h*	$\beta^-(\gamma)$	1,1				
	62	**Sm**	**25**	**87**		γ	0,25				
150	**60**	**Nd**	**30**	**90**				-356 ± 21	13285	1236,6	8,244
	62	**Sm**	**26**	**88**							
151	62	*Sm*	27	89	≈ 20 *a*						
	63	**Eu**	**25**	**88**							
152	**62**	**Sm**	**28**	**90**	3,0 . 10^{11} *a*	α	2,4				
	63	*Eu*	26	89	≈ 5 *a*						
	63	*Eu*(*)	26	89	9,2 *h*	$\beta^-(\gamma)$	1,885 ± 0,012				
	64	**Gd**	**24**	**88**		γ	0,123 ± 0,001				
						γ	0,163 ± 0,001				
						γ	0,725 ± 0,003				
153	61	*Pm*	31	92	< 5 *m*	β^-					
	62	*Sm*	29	91	47 ± 1 *h*	$\beta^-(\gamma)$	0,73				
	63	**Eu**	**27**	**90**		γ	0,694				
						γ	0,103				
	64	*Gd*	25	89	> 72 *d*						
154	**62**	**Sm**	**30**	**92**							
	63	*Eu*	28	91	22 *a*	$\beta^-(\gamma)$	0,93				
						$\beta^-(\gamma)$	0,62				
	64	**Gd**	**26**	**90**		γ	0,1224				
						γ	0,2473				
						γ	0,2860				
						γ	0,3428				
						γ	0,4078				
						γ	1,23				
155	62	*Sm*	(31)	(93)	21 *m*	$\beta^-(\gamma)$	1,8				
	63	*Eu*	29	92	2 ... 3 *a*	γ	≈ 0,3				
						$\beta^-(\gamma)$	0,23				
	64	**Gd**	**27**	**91**		γ	0,0844	-426	13770	1281,8	8,270
156	61	*Pm*	(34)	(95)	< 5 *m*	β^-					
	62	*Sm*	(32)	(94)	≈ 10 *h*	β^-	≈ 0,8				
	63	*Eu*	30	93	15,4 *d*	40 % β^-	2,4				
						60 % $\beta^-(\gamma_1)$	≈ 0,5				
	64	**Gd**	**28**	**92**		γ_1	2,0	-432	13865	1290,6	8,273

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|𝓔|		|𝓔|/A
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
157	63	*Eu*	(31)	(94)	15,4 *h*	25 % β^-	≈ 1,7				
						75 % $\beta^-(\gamma)$	≈ 1,0				
	64	**Gd**	**29**	**93**		γ	0,2	-432	13954	1298,9	8,273
						γ	0,6				
158	**64**	**Gd**	**30**	**94**				-303	13916	1295,4	8,199
	66	**Dy**	**26**	**92**							
159	**65**	**Tb**	**29**	**94**							
160	**64**	**Gd**	**32**	**96**				-307	14098	1312,3	8,202
	65	*Tb*	30	95	3,9 *h*	β^-					
	65	*Tb*(*)	30	95	72 *d*	$\beta^-(\gamma)$	0,70				
	66	**Dy**	**28**	**94**		γ	0,0856				
						γ	0,1947				
						γ	0,2132				
						γ	0,2980				
						γ	1,10				
161	**66**	**Dy**	**29**	**95**							
162	**66**	**Dy**	**30**	**96**							
	68	**Er**	**26**	**94**							
163	**66**	**Dy**	**31**	**97**							
164	**66**	**Dy**	**32**	**98**							
	68	**Er**	**28**	**96**							
165	66	*Dy*	33	99	156 ± 3 *m*	$\beta^-(\gamma_1)$	1,24 ± 0,03				
						$\beta^-(\gamma_2)$	0,88				
						$\beta^-(\gamma_3)$	0,42				
	66	*Dy**	33	99	1,25 *m*	*I.T.* γ	0,18				
	67	**Ho**	**31**	**98**		γ_1	0,0914				
						γ_2	0,36				
						γ_3	0,76				
166	67	*Ho*	32	99	27,3 ± 0,5 *h*	$\beta^-(\gamma)$	1,8				
	68	**Er**	**30**	**98**							
167	**68**	**Er**	**31**	**99**							
168	**68**	**Er**	**32**	**100**							
	70	**Yb**	**28**	**98**							
169	68	*Er*	(33)	(101)	9,4 ± 0,2 *d*	β^-(no γ)	0,33 ± 0,01				
	69	**Tm**	**31**	**100**		γ	0,4				
	69	*Tm**	(31)	(100)	10^{-6} *s*	*I.T.* γ'	0,2				
	70	*Yb*	(29)	(99)	33 ± 1,5 *d*	$K(\gamma\gamma')$					
170	**68**	**Er**	**34**	**102**							
	69	*Tm*	32	101	127 ± 5 *d*	β^-(no γ)	1,1				
	70	**Yb**	**30**	**100**							

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10^{-4} MU	$\lvert\delta\rvert$ 10^{-4} MU	$\lvert\delta\rvert$ MeV	$\lvert\delta\rvert/A$ MeV
1	2	3	4	5	6	7	8	9	10	11	12
171	68	Er	(35)	(103)	7,5 h	6 % β^- (no γ)	1,49 ± 0,05				
						72 % β^- ($\gamma_2\gamma_1$)	1,05 ± 0,03				
						22 % β^- (γ_3)	0,67 ± 0,03				
	69	Tm	(33)	(102)	500 ± 100 d	γ_3	0,805 ± 0,025				
	69	Tm*	(33)	(102)	$2{,}5 \cdot 10^{-6}$ s	γ_2	0,305 ± 0,010				
						I.T. γ_1	0,113 ± 0,005				
	70	Yb	31	101							
172	70	Yb	32	102							
173	70	Yb	33	103							
174	70	Yb	34	104							
	72	Hf	30	102							
175	70	Yb	35	105	4,2 ± 0,2 d	β^-	0,45 ± 0,07				
						β^- (γ)	0,13				
	71	Lu	33	104		γ	0,35				
176	70	Yb	36	106							
	71	Lu	34	105	$2{,}4 \cdot 10^{10}$ a	K					
						β^- (γ)	0,215 ± 0,015				
	71	Lu*	(34)	(105)	3,67 ± 0,03 h	β^-	1,25 ± 0,07				
	72	Hf	32	104		γ	0,260				
177	70	Yb	(37)	(107)	1,9 ± 0,2 h	β^-	1,15 ± 0,10				
	71	Lu	35	106	6,6 ± 0,1 d	β^-	0,440				
	72	Hf	33	105							
178	72	Hf	34	106							
179	72	Hf	35	107							
180	72	Hf	36	108							
	73	Ta	34	107	8,2 ± 0,2 h	K (γ)					
	73	Ta(*)	34	107	17 m						
	74	W	32	106							
181	72	Hf	37	109	55 ± 7 d	β^- ($\gamma\gamma'$)	0,63 ± 0,03				
	73	Ta	35	108		γ	0,3455				
						γ	0,4787				
						γ	0,600				
	73	Ta*	35	108	$2{,}2 \cdot 10^{-5}$ s	γ''	1,83				
						I.T. γ'	0,1325				
	74	W	(33)	(107)	140 d	K ($\gamma''\gamma'$)					
182	73	Ta	36	109	99 ± 1 d	β^- (γ)	0,52 ± 0,03				
	73	Ta*	(36)	(109)	16,2 ± 0,5 m	β^- (γ)					
	74	W	34	108		γ	0,15				
						γ	0,22				
						γ	1,13				
						γ	1,22				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation MeV	W—A 10⁻⁴ MU	|δ| 10⁻⁴ MU	|δ| MeV	|δ|/A MeV
1	2	3	4	5	6	7	8	9	10	11	12
183	74	W	35	109							
184	74	W	36	110		γ	0,85 ± 0,1				
	75	Re	(34)	(109)	54 ± 2 d	$K(\gamma)$					
	76	Os	32	108							
185	74	W	(37)	(111)	73,2 ± 0,5 d	β^- (no γ)	0,430 ± 0,010				
	75	Re	35	110		γ	0,75				
	76	Os	(33)	(109)	94,7 ± 2 d	$K(\gamma)$					
186	74	W	38	112							
	75	Re	36	111	92,8 ± 0,2 h	β^- (no γ)	1,07				
	76	Os	34	110							
187	74	W	(39)	(113)	24,1 ± 0,1 h	60 % β^- (no γ)	1,40 ± 0,05				
						40 % β^- (γ)	0,55				
	75	Re	37	112	$4 \pm 1 . 10^{12}$ a	γ	0,086				
						γ	0,101				
						γ	0,135				
						γ	0,48				
						γ	0,57				
						γ	0,69				
						γ	0,79				
						γ	0,86				
						β^-	0,043				
	75	Re*	37	112	10^{-6} s	I.T. γ	0,135 ?				
	76	Os	35	111							
188	75	Re	38	113	18,9 ± 0,2 h	β^- (γ)	2,05				
	76	Os	36	112		γ	0,16				
						γ	0,48				
						γ	0,64				
						γ	0,94				
						γ	1,43				
189	76	Os	37	113							
190	76	Os	38	114		γ	0,25	238 ± 63	16139	1502,3	7,907
	77	Ir	(36)	(113)	10,7 ± 0,3 d	$K(\gamma)$					
	78	Pt	(34)	(112)	long						
191	76	Os	39	115	31,8 ± 0,2 h	β^- (γ)	0,95				
	77	Ir	37	114		γ	1,17	246 ± 64	16212	1509,1	7,901
						γ'	0,57				
	78	Pt	(35)	(113)	3,00 ± 0,02 d	$K?(\gamma')$					
192	76	Os	40	116				240 ± 63	16316	1518,8	7,910
	77	Ir	38	115	75 ± 3 d	β^- (γ)	0,59 ± 0,01				
	78	Pt	36	114		γ	0,307				
						γ	0,467				
						γ	0,603				

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W−A	|𝓔|		|𝓔|/A
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV
1	2	3	4	5	6	7	8	9	10	11	12
193	76	*Os*	(41)	(117)	16,1 ± 0,2 *d*	$\beta^-(\gamma)$	0,142 ± 0,003				
	77	**Ir**	**39**	**116**		γ	0,039 ± 0,001	249 ± 64	16388	1525,5	7,904
						γ	0,1291				
						γ'	0,17				
						γ'	1,7				
	78	*Pt*	(37)	(115)	4,33 ± 0,02 *d*	$K(\gamma')$					
	79	*Au*	(35)	(114)	15,8 ± 0,3 *h*	K					
194	77	*Ir*	40	117	19,0 ± 0,2 *h*	80% β^-	2,07 ± 0,03				
						20% $\beta^-(\gamma_1\gamma_2)$					
	78	**Pt**	**38**	**116**		γ_1	0,38	239 ± 80	16479	1534,0	7,907
						γ_2	1,65				
195	**78**	**Pt**	**39**	**117**		γ	0,17	273 ± 78	16535	1539,2	7,893
						γ	1,7				
	79	*Au*	(37)	(116)	195 ± 5 *d*	$K(\gamma)$					
196	**78**	**Pt**	**40**	**118**		γ'	0,139	235 ± 80	16662	1551,0	7,913
						γ'	0,358				
	79	*Au*	(38)	(117)	13 *h*	β^-					
	79	*Au*(*)	(38)	(117)	5,55 ± 0,12 *d*	6% $\beta^-(\gamma_1)$	0,43				
						24% $\beta^-(\gamma_2)$	0,27				
						70% $K(\gamma')$					
	80	**Hg**	**36**	**116**		γ_1	0,173				
						γ_2	0,334				
197	78	*Pt*	(41)	(119)	19 ± 2 *h*	β^-	0,68				
	78	*Pt**	(41)	(119)	3,3 *d*	$\beta^-(\gamma)$					
	79	**Au**	**39**	**118**		γ_4	0,127	400 ± 40	16579	1543,3	7,834
						γ_5	0,159				
	79	*Au**	39	118	7,4 ± 0,2 *s*	$I.T.(\gamma_1)$ γ_1	0,38				
						$I.T.(\gamma_2\gamma_3)$ γ_2	0,25				
						γ_3	0,077				
	80	*Hg*	37	117	64 *h*	$K(\gamma_3)$					
	80	*Hg**	37	117	23 *h*	$K(\gamma_4\gamma_5)$					
198	**78**	**Pt**	**42**	**120**				317 ± 80	16759	1560,0	7,879
	79	*Au*	40	119	2,76 ± 0,02 *d*	K					
						85% $\beta^-(\gamma_1)$	0,970 ± 0,005				
						15% $\beta^-(\gamma_2\gamma_3\gamma_1)$	0,605				
	80	**Hg**	**38**	**118**		γ_1	0,4112				
						γ_2	0,208				
						γ_3	0,157				
199	78	*Pt*	43	121	31 *m*	β^-	1,8				
	79	*Au*	41	120	3,3 *d*	$\beta^-(\gamma)$	0,38				
	80	**Hg**	**39**	**119**		γ	0,18				
200	**80**	**Hg**	**40**	**120**							
201	**80**	**Hg**	**41**	**121**							
202	**80**	**Hg**	**42**	**122**		γ	0,40				
	81	*Tl*	(40)	(121)	11,8 ± 1,2 *d*	$K(\gamma)$					

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	$\|\mathcal{E}\|$		$\|\mathcal{E}\|/A$	Sy bo
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
203	80	*Hg*	(43)	(123)	51,5 *d*	β^-	0,46					
						$\beta^-(\gamma)$	0,11					
	81	**Tl**	**41**	**122**		γ	0,270	438	17061	1588,1	7,823	
						γ'	0,470					
	82	*Pb*	(39)	(121)	52 ± 0,5 *h*	$K(\gamma\gamma')$						
	82	*Pb*(*)	(39)	(121)	10,25 *m*	β^+	1,66					
204	**80**	**Hg**	**44**	**124**								
	81	*Tl*	(42)	(123)	3,5 *a*	β^-(no γ)	0,87					
	82	**Pb**	**40**	**122**		γ'	≈0,2	432	17148	1596,2	7,825	
	82	*Pb**	40	122	68 ± 2 *m*	*I.T.* γ	0,90					
	83	*Bi*	38	121	12 *h*	$K(\gamma\gamma')$						
205	80	*Hg*	45	125	5,5 *m*	β^-	1,62					
	81	**Tl**	**43**	**124**				449	17229	1603,8	7,823	
206	81	*Tl*	44	125	4,23 ± 0,03 *m*	β^-(no γ)	1,77	461,5	17305,9	1611,35	7,822	*Ral*
	82	**Pb**	**42**	**124**		γ	1,1	443,1	17316,1	1612,30	7,827	**Ra(**
	83	*Bi*	40	123	6.4 *d*	γ	0,4					
						γ	0.8					
						$K(\gamma)$						
	84	*Po*	(38)	(122)	9 *d*	90 % $K(\gamma)$						
						10 % α	5,3					
207	81	*Tl*	45	126	4,76 ± 0.02 *m*	99,5 % β^-	1,47	478,7	17378,1	1618.07	7,817	*Ac*
						0,5 % $\beta^-(\gamma)$	≈ 0,4					
	82	**Pb**	**43**	**125**		γ	0,87	462.9	17385,8	1618,79	7,820	**Ac.**
	84	*Po*	(39)	(123)	5,7 *h*	99,99 % $K(\gamma')$						
						0,01 % α	5,2					
208	81	*Tl*	46	127	3,1 *m*	10 % $\beta^-(\gamma_2\gamma_1)$	1,820	525,9	17420.4	1622,01	7,798	*Th*
						18 % $\beta^-(\gamma_3\gamma_1)$	1,586					
						72 % $\beta^-(\gamma_4\gamma_1)$	1,514					
	82	**Pb**	**44**	**126**		γ_1	2,62	472,0	17466,1	1626,27	7,819	**Th**
						γ_2	0.277					
						γ_3	0,511					
						γ_4	0,583					
	83	*Bi*	(42)	(125)	long			508	17422	1622,2	7,799	
	84	**Po**	**(40)**	**(124)**	≈ 3 *a*	α	5,24					
209	81	*Tl*	47	128	< 1 *h*	β^-		562,1	17473,6	1626,97	7,785	
	82	*Pb*	45	127	3,30 ± 0,03 *h*	β^-	0,68 ± 0.03	521,3	17506,3	1630,01	7,799	
	83	**Bi**	**43**	**126**				514,0	17505,4	1629,93	7,799	
210	81	*Tl*	48	129	1,32 *m*	$\beta^-(\gamma?)$	1,80	625.5	17500,6	1629,48	7,759	*Re*
	82	*Pb*	46	128	22 *a*	$\beta^-(\gamma)$	0,035	553,6	17563,4	1635,33	7,787	*Re*
	83	*Bi*	44	127	5,0 *d*	γ	0,012	553,1	17555,8	1634.62	7,784	*Re*
						β^-(no γ)	1,22					
						$5 . 10^{-5}$ % α						
	84	**Po**	**42**	**126**	140 *d*	α	5,403	539,9	17560,9	1635,09	7,786	**R**

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|δ|		|δ|/A	Symbol
							MeV	10^{-4} MU	10^{-4} MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
211	82	*Pb*	47	129	36,1 ± 0,2 *m*	80 % β^-	1,40	605,0	17601,5	1638,87	7,767	*AcB*
						20 % $\beta^-(\gamma_1)$	0,571					
	83	*Bi*	45	128	2,16 *m*	γ_1	0,829	589,9	17608,4	1639,52	7,770	*AcC*
						0,32 % $\beta^-(\gamma)$						
						99,68 % α	6,739					
	84	**Po**	**43**	**127**	$5 \cdot 10^{-3}$ *s*	α	7,581	583,0	17607,2	1639,40	7,769	**AcC′**
	85	*At*	41	126	7,5 *h*	40 % *K*						
						60 % α	6,05					
212	82	*Pb*	48	130	10,6 *h*	$\beta^-(\gamma_1)$	0,362	637,5	17658,4	1644,17	7,756	*ThB*
	83	*Bi*	46	129	60,5 *m*	γ_1	0,238	631,1	17656,7	1644,01	7,755	*ThC*
						66,3 % β^-	2,25					
						33,7 % α	6,200					
	84	**Po**	**44**	**128**	$3 \cdot 10^{-7}$ *s*	α	8,947	606,9	17672,8	1645,51	7,762	**ThC′**
213	83	*Bi*	47	130	46 *m*	98 % β^-	≈1,25	664,8	17712,4	1649,20	7,743	
						2 % α	5,97					
	84	**Po**	**45**	**129**	$4,4 \cdot 10^{-6}$ *s*	α	8,496	651,3	17717,8	1649,70	7,745	
214	82	*Pb*	50	132	26,8 *m*	$\beta^-(\gamma_1)$	0,65	726,8	17748,1	1652,52	7,722	*RaB*
	83	*Bi*	48	131	19,7 *m*	γ_1	0,294	716,8	17749,9	1652,69	7,723	*RaC*
						99,96 % $\beta^-(\gamma_2)$	3,15					
						0,04 % α	5,612					
	84	**Po**	**46**	**130**	$1,5 \cdot 10^{-4}$ *s*	γ_2	0,607	676,4	17782,1	1655,69	7,737	**RaC′**
						α	7,829					
215	84	*Po*	47	131	$1,83 \cdot 10^{-3}$ *s*	$5 \cdot 10^{-4}$ % β^-		724,4	17823,6	1659,55	7,718	*AcA*
						α	7,509					
	85	**At**	**45**	**130**	≈ 10^{-4} *s*	α	8,15	716,4	17823,5	1659,56	7,719	
216	**84**	**Po**	**48**	**132**	0,158 *s*	α	6,904	750,1	17887,4	1665,49	7,711	**ThA**
	85	**At**	**46**	**131**	10^{-3} *s*	α	7,94	755,1	17874,2	1664,26	7,705	
217	**85**	**At**	**47**	**132**	0,020 *s*	α	7,157	780,5	17938,3	1670,23	7,697	
218	**84**	**Po**	**50**	**134**	3,05 *m*	α	6,1124	831,1	17985,2	1674,60	7,682	**RaA**
	86	**Em**	**46**	**132**	0,019 ± 0,002 *s*	α	7,25	793,1	18007,0	1676,63	7,691	
219	**86**	**Em**	**47**	**133**	3,92 *s*	α	6,953	837,8	18051,7	1680,79	7,675	**An**
	87	**Fr**	**45**	**132**	≈ 10^{-4} *s*	α	7,44	834,9	18046,5	1680,31	7,673	
220	**86**	**Em**	**48**	**134**	54,5 *s*	α	6,400	857,9	18121,1	1687,25	7,669	**Tn**
	87	**Fr**	**46**	**133**	≈ 30 *s*	α	6,81	867,3	18103,6	1685,64	7,662	
221	**87**	**Fr**	**47**	**134**	4,8 *m*	α	6,43	888,3	18172,0	1691,99	7,656	
222	**86**	**Em**	**50**	**136**	3,825 *d*	α	5,5887	929,9	18227,9	1697,20	7,645	**Rn**
	88	**Ra**	**46**	**134**	38,0 *s*	α	6,63	903,1	18238,5	1698,19	7,650	
223	87	*Fr*	(49)	(136)	21 *m*	β^-	1,20	952,3	18286,9	1702,69	7,635	*AcK*
	88	**Ra**	**47**	**135**	11,2 *d*	α	5,823	939,1	18291,9	1703,16	7,639	**AcX**
	89	**Ac**	**45**	**134**	≈2 *m*	α	6,76	947,5	18276,5	1701,74	7,630	

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|δ|		|δ|/A	Symbol
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
224	88	Ra	48	136	3,64 d	α	5,7858	958,8	18361,6	1709,65	7,632	ThX
	89	Ac	46	135	≈2,5 h	10 % α	6,28	975,5	18338,8	1707,55	7,623	
						90 % K						
225	88	Ra	49	137	14,8 d	β^- (γ?)	≈0,05	998,7	18411,3	1714,27	7,619	
	89	Ac	47	136	10,0 d	α	5,906	994,2	18407,6	1713,93	7,617	
226	88	Ra	50	138	1590 a	α	4,879	1021,1	18478,3	1720,51	7,613	Ra
	90	Th	46	136	30,9 m	α	6,41	1010,6	18472,5	1719,97	7,610	
227	89	Ac	49	138	13,5 a	99 % β^-	0,22	1046,5	18534,2	1725,72	7,602	Ac
						1 % α	5,10					
	90	Th	47	137	18,9 d	α	6,159	1045,0	18528,5	1725,19	7,600	RdAc
	91	Pa	45	136	38 m	83 % α	6,58	1056,0	18508,5	1723,35	7,592	
						17 % K						
228	88	Ra	52	140	6,7 a	β^-	0,053	1084,7	18593,6	1731,25	7,593	$MsTh_1$
	89	Ac	50	139	6,13 h	β^- (γ)	1,55	1084,0	18586,1	1730,55	7,590	$MsTh_2$
	90	Th	48	138	1,90 a	γ	0,970	1056,9	18605,1	1732,32	7,598	RdTh
						α	5,517					
	91	Pa	46	137	22 h	2 % α	6,20	1079,0	18575,0	1729,54	7,586	
						98 % K						
229	90	Th	49	139	7 . 10^3 a	α	4,94	1094,1	18657,3	1737,18	7,585	
	91	Pa	(47)	(138)		K						
						α						
230	90	Th	50	140	8,3 . 10^4 a	α	4,76	1110,9	18729,9	1743,94	7,582	Io
	91	Pa	48	139	17 d	β^-						
	92	U	46	138	20,8 d	α	5,96	1113,4	18711,2	1742,20	7,567	
231	90	Th	51	141	24,6 h	β^-	≈ 0,2	1150,4	18780,0	1748,60	7,570	UY
	91	Pa	49	140	3,2 . 10^4 a	α	5,142	1141,4	18780,8	1748,68	7,570	Pa
232	90	Th	52	142	1,39 . 10^{10} a	α	4,28	1169,4	18850,4	1755,16	7,565	Th
	91	Pa	50	141	1,33 d	β^-						
	92	U	48	140	30 a	α	5,41	1153,6	18849,8	1755,10	7,565	
233	90	Th	53	143	23 m	β^-						
	91	Pa	51	142	27,4 d	β^- (γ)	0,23					
	92	U	49	141	1,63 . 10^5 a	γ	0,084	1185,6	18907,3	1760,46	7,556	
						γ	0,298					
						γ	0,338					
						γ	0,310					
						α	4,915					

A	Z	Symbol	n	N	Half-life	Decay	Energy of radiation	W—A	|$\mathcal{E}$|		|$\mathcal{E}$|/A	Symbol
							MeV	10^{-4} MU	10^{-4}MU	MeV	MeV	
1	2	3	4	5	6	7	8	9	10	11	12	13
234	90	*Th*	54	144	24,1 *d*	β^-	0,300	1229,5	18969,2	1766,22	7,548	UX_1
	91	*Pa*	52	143	6,7 *h*	10 % $\beta^-(\gamma')$	1,2	1222,0	18968,6	1766,16	7,548	*UZ*
						90 % $\beta^-(\gamma'\gamma'')$	0,45 ± 0,03					
	91	*Pa**	52	143	1,14 *m*	0,15 % *I.T.* γ	0,394 ± 0,005	1227,2	18964,4	1765,77	7,546	UX_2
						98 % β^-	2,32					
						1 % $\beta^-(\gamma_1)$	1,54					
						1 % $\beta^-(\gamma_2)$	1,50					
						0,2 % $\beta^-(\gamma_3)$						
	92	**U**	**50**	**142**	2,30 . 10^5 *a*	$\gamma' \approx \gamma''$	0,80 ± 0,05	1201,2	18981,2	1767,34	7,553	**U II**
						γ_1	0,782					
						γ_2	0,822					
						γ_3	1,5					
						α	4,79					
	93	*Np*	48	141	4,4 *d*	$K(\gamma)$						
235	**92**	U	**51**	**143**	7,07 . 10^8 *a*	α	4,59	1238,4	19033,4	1772,20	7,541	**AcU**
	93	*Np*	49	142	240 *d*	$K(\gamma)$						
236	93	*Np*	50	143	20 *h*	β^-						
	94	**Pu**	**48**	**142**		α	5,85	1255,3	19089,7	1777,44	7,531	
237	92	*U*	53	145	7,0 ± 0,2 *d*	$\beta^-(\gamma)$	0,26					
	93	**Np**	**51**	**144**	2,25 . 10^6 *a*	γ	0,5					
						α	4,85					
	94	*Pu*	49	143		*K*						
238	**92**	**U**	**54**	**146**	4,51 . 10^9 *a*	α	4,21	1313,5	19226,7	1790,20	7,522	**U I**
	93	*Np*	52	145	2,0 *d*	$\beta^-(\gamma)$	1,35					
	94	**Pu**	**50**	**144**	60 *a*	γ	0,1	1300,4	19223,5	1789,90	7,521	
						γ	0,25					
						γ	1,3					
						α	5,590 ± 0,005					
239	92	*U*	55	147	23,5 *m*	$\beta^-(\gamma_1)$	1,12	1359,3	19270,4	1794,26	7,507	
	93	*Np*	53	146	2,3 *d*	γ_1	0,073	1346,3	19275,2	1794,71	7,509	
						β^-	1,179					
						$\beta^-(\gamma_1\gamma_2)$	0,676					
						$\beta^-(\gamma_1...\gamma_4)$	0,403					
						$\beta^-(\gamma_1...\gamma_6)$	0,288					
	94	**Pu**	**51**	**145**	2,41 . 10^4 *a*	γ_1	0,275	1333,6	19279,8	1795,14	7,511	
						γ_2	0,227					
						γ_3	0,206					
						γ_4	0,067					
						γ_5	0,061					
						γ_6	0,057					
						α	5,227 ± 0,005					
240	**96**	**Cm**	**48**	**144**	30 *d*	α	6,36	1362,4	19324,2	1799,27	7,496	
241	94	*Pu*	53	147	long	β^-						
						α						
	95	**Am**	**51**	**146**	500 *a*	α	5,56					
	96	*Cm*	(49)	(145)		*K*						
242	95	*Am*	52	147	18 *h*	β^-						
	96	**Cm**	**50**	**146**	150 *d*	α	6,1	1403,5	19462,0	1812,10	7,488	

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CORRIGENDUM.

Diagram IV. Third line. Read $^{187}_{76}Os$ instead of $^{187}_{67}Os$.

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